New approach for the solution of the Dirac equation

An effective approach for the solution of the three dimensional Dirac equation for local interactions, which we have introduced recently, is reviewed. The merit of the approach is in producing Schrödinger-like equation for the spinor components that could simply be solved by correspondence with well-known exactly solvable non-relativistic problems. Taking the nonrelativistic limit will reproduce the nonrelativistic problem. The approach has been used successfully in establishing the relativistic extension of all known classes of exactly solvable non-relativistic problems. These include the Coulomb, Oscillator, Scarf, Pöschl-Teller, Woods-Saxon, etc.

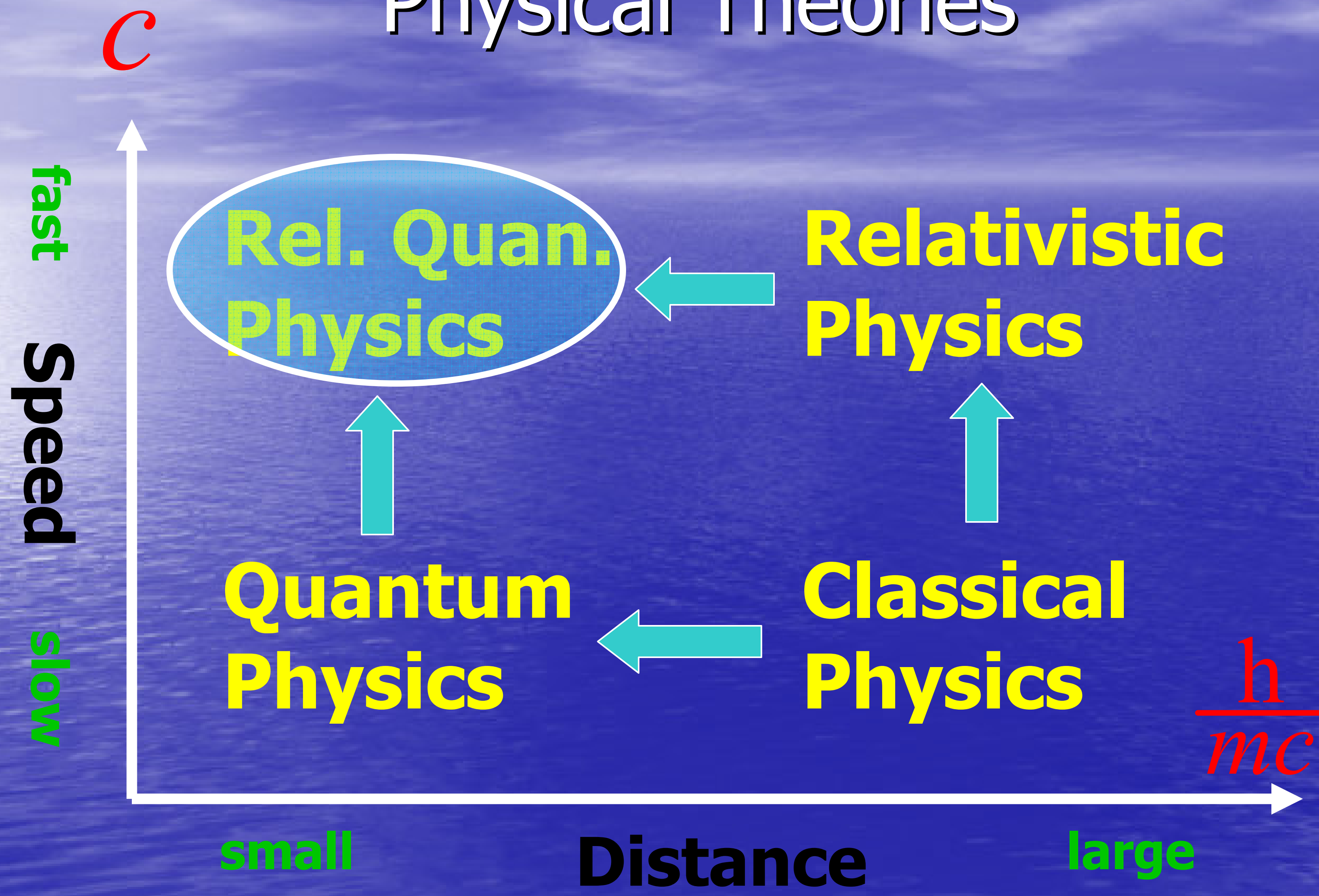
- In recent decades: remarkable progress in human knowledge and technology. Large collaborations + Lots of money
- Examples in Physics: unprecedented discoveries and findings (e.g. materials, particles, information theory, etc.)

- Working as individuals on simple problems un-rewarding.
- Exact solutions are (debatable) “trivial”!

- However, exact solutions are important for conceptual understanding of Physics.
- Checking & improving models and numerical methods for solving complicated problems.
- Analytic solutions or approximations of some realistic problems.

- Nonrelativistic QM: Lots of work, many authors (SUSYQM + potential algebra + PCT + …) → "shape invariant Potentials"
- Exactly, conditionally exactly, quasi exactly solvable problems.

Physical Theories



	$V(r)$	E_n
Coulomb	Z/r	$-\frac{1}{2}\left(\frac{Z}{n+1+1}\right)^2$
Oscillator	$\frac{1}{2}\omega^4 r^2$	$\omega^2\left(2n+1+3/2\right)$
Morse	$-\lambda\left(2A+\lambda\right)e^{-\lambda r}+2\lambda^2 e^{-2\lambda r}$	$-\frac{\lambda^2}{2}\left(A/\lambda-n\right)^2$
Hulthén	$-Ae^{-\lambda r}/(1-e^{-\lambda r})$	$-\frac{\lambda^2}{8}\left(\frac{2A/\lambda^2}{n+1}-n-1\right)^2$
Rosen-Morse I	$B\tanh(\lambda r)-\frac{A}{2}(A+\lambda)\frac{1}{\cosh^2(\lambda r)}$	$-\frac{\lambda^2}{2}\left(A/\lambda-n\right)^2-\frac{(B/\lambda)^2}{2\left(A/\lambda-n\right)^2}$
Eckart	$B\coth(\lambda r)+\frac{A}{2}(A-\lambda)\frac{1}{\sinh^2(\lambda r)}$	$-\frac{\lambda^2}{2}\left(A/\lambda+n\right)^2-\frac{(B/\lambda)^2}{2\left(A/\lambda+n\right)^2}$
Rosen-Morse II	$\lambda\left(A-\frac{\lambda}{2}\right)\frac{\cosh(\lambda r)}{\sinh(\lambda r)^2}+\frac{1}{2}\left(A-\frac{\lambda}{2}\right)^2\frac{1}{\sinh(\lambda r)^2}$	$-\frac{\lambda^2}{2}\left(A/\lambda+n\right)^2$
Scarf	$\lambda\left(A+\frac{\lambda}{2}\right)\frac{\sinh(\lambda r)}{\cosh(\lambda r)^2}-\frac{1}{2}\left[A^2-(\lambda/2)^2+\lambda A\right]\frac{1}{\cosh(\lambda r)^2}$	$-\frac{\lambda^2}{2}\left(A/\lambda-n\right)^2$
Pöschl-Teller	$\frac{B}{2}(B+\lambda)\frac{1}{\sinh(\lambda r)^2}-\frac{A}{2}(A+\lambda)\frac{1}{\cosh^2(\lambda r)}$	$-\frac{\lambda^2}{2}\left(\frac{A+B}{\lambda}-2n\right)^2$

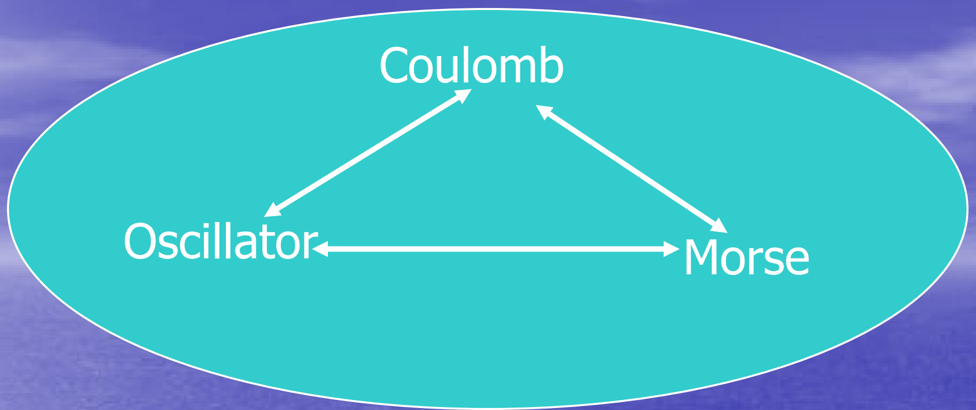
- Relativistic extension, however, remained for a long time only partially developed.
- Exact solutions of Dirac equation, up until 1989, only for the Dirac-Coulomb problem.
- In 1989: Moshinsky and Szczepaniak solved the Dirac-Oscillator (linear in r).

- Nonrelativistic: Coulomb + Oscillator + Morse = SO(2,1)
- Relativistic: Dirac-Coulomb + Dirac-Oscillator + ??? = ???

$1/r$

r^2

$\left(e^{-r}-1\right)^2$



SO(2,1)

PCT

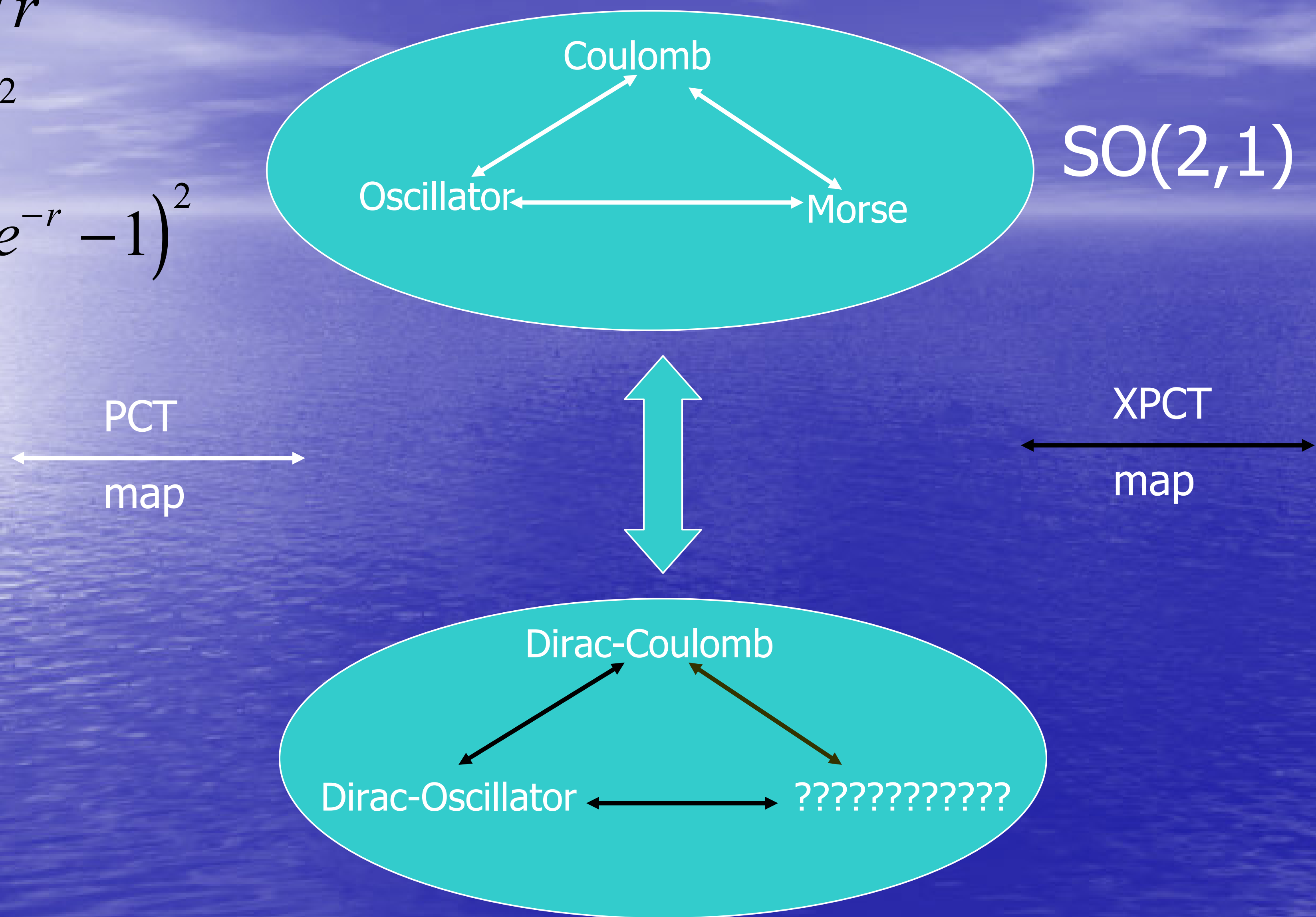
$\longleftrightarrow$

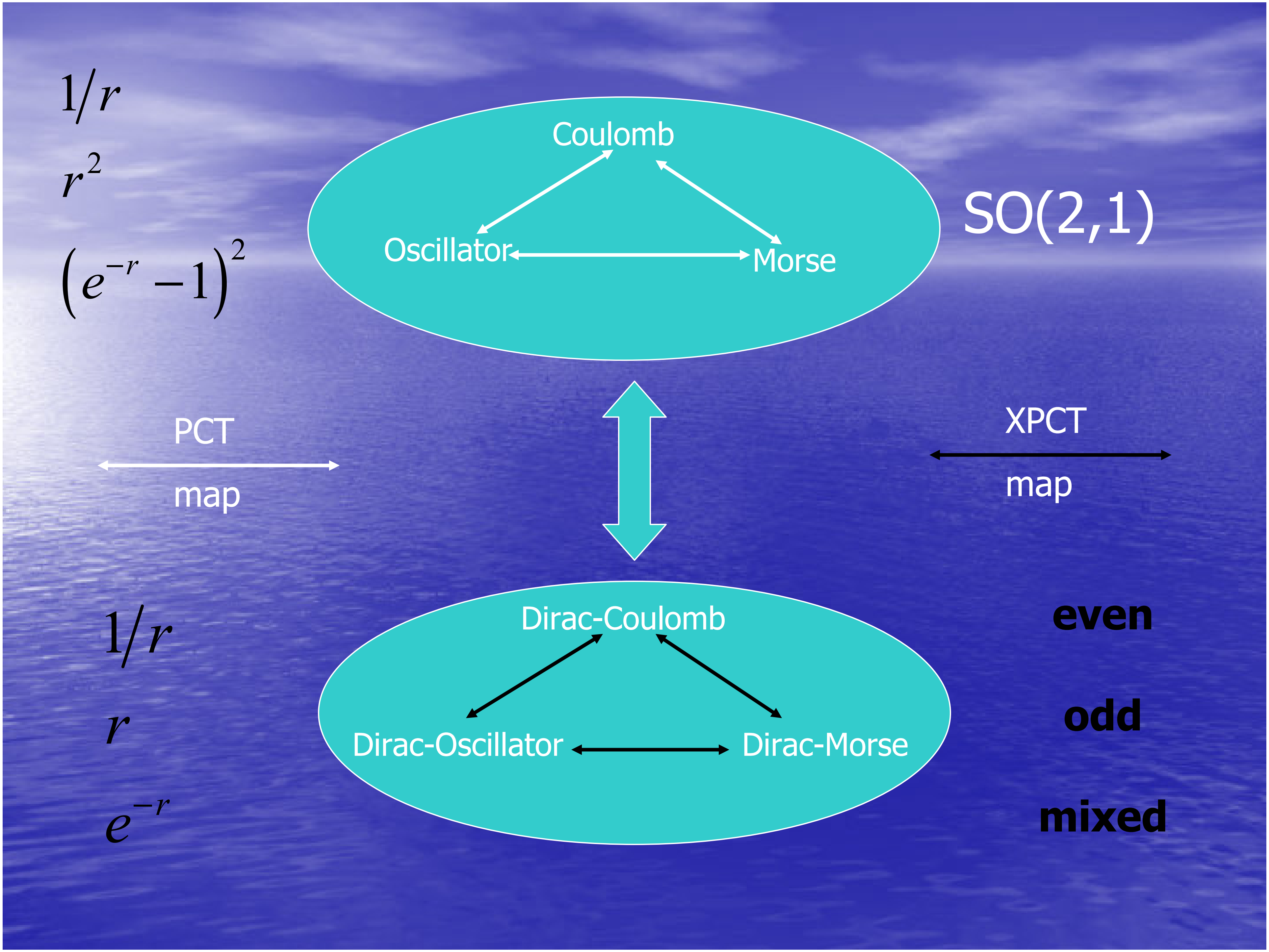
map

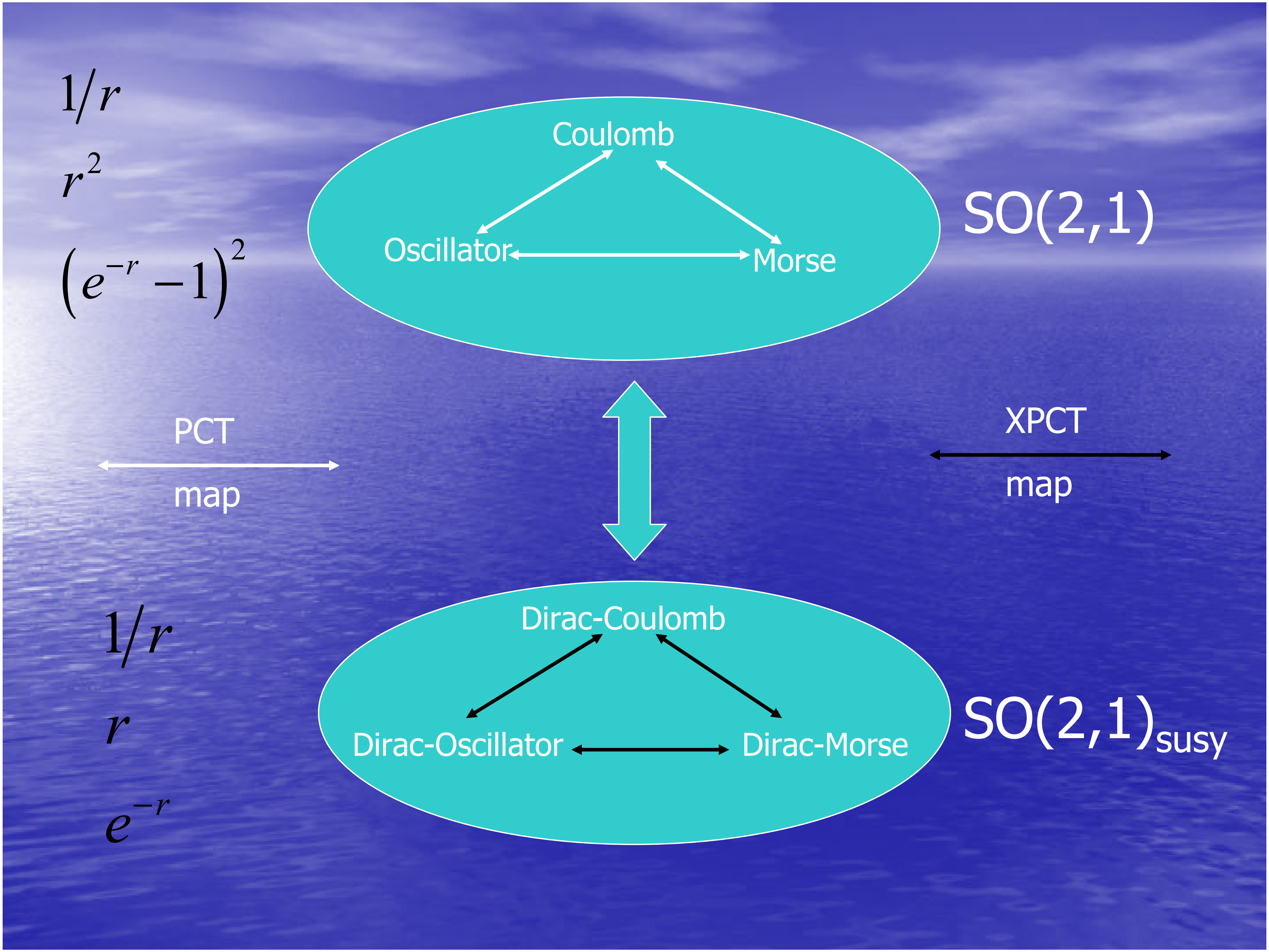
$$1/r$$

$$r^2$$

$$\left(e^{-r}-1\right)^2$$




$1/r$
r^2
$\left(e^{-r}-1\right)^2$
Coulomb
Oscillator
Morse
SO(2,1)
PCT
map
XPCT
map
$1/r$
r
e^{-r}
Dirac-Coulomb
Dirac-Oscillator
Dirac-Morse
even
odd
mixed


$1/r$
r^2
$\left(e^{-r}-1\right)^2$
Coulomb
Oscillator
Morse
SO(2,1)
PCT
map
XPCT
map
$1/r$
r
e^{-r}
Dirac-Coulomb
Dirac-Oscillator
Dirac-Morse
$SO(2,1)_{susy}$

- Confluent Hypergeometric (oscillator) class:
  Dirac-Coulomb, Dirac-Oscillator, Dirac-Morse
- Hypergeometric (Rosen-Morse) class:
  Dirac-Scarf, Dirac-Eckart, Dirac-Hulthén, Dirac-Rosen-Morse I, Dirac-Rosen-Morse II, Dirac-Pöschl-Teller
- No-name class:
  Dirac-Woods-Saxon, ????

	$V(r)$	$W(r)$	$\pm\mathrm{D}\,\eta$	Parameters Relation
Dirac-Coulomb	Z/r	0	$\sin^{-1}(\mathrm{D}Z/\kappa)$	$\kappa^2=\gamma^2+\mathrm{D}^2Z^2$
Dirac-Oscillator	0	$\omega^2 r$	0	$\kappa=1,-1-1$
Dirac-Morse	$-Be^{-\mu r}$	$-Ae^{-\mu r}+1/r$	$\sin^{-1}(\mathrm{D}B/A)$	$A^2=\mu^2+\mathrm{D}^2B^2$
Dirac-Rosen-Morse I	$B\tanh(\mu r)$	$A\tanh(\mu r)+1/r$	$\sin^{-1}(\mathrm{D}B/A)$	$A^2=\mu^2+\mathrm{D}^2B^2$
Dirac-Eckart	$B\coth(\mu r)$	$A\coth(\mu r)+1/r$	$\sin^{-1}(\mathrm{D}B/A)$	$A^2=\mu^2+\mathrm{D}^2B^2$
Dirac-Rosen-Morse II	0	$A\coth(\mu r)-C\,\mathrm{csch}(\mu r)+1/r$	0	———
Dirac-Scarf	0	$A\tanh(\mu r)+C\,\mathrm{sech}(\mu r)+1/r$	0	———
Dirac-Pöschl-Teller	0	$-A\tanh(\mu r)-C\coth(\mu r)+1/r$	0	———
Dirac-Woods-Saxon	$\frac{-B}{1+e^{\mu(r-R)}}$	$\frac{A}{1+e^{\mu(r-R)}}+1/r$	$-\sin^{-1}(\mathrm{D}B/A)$	$A^2=\mu^2+\mathrm{D}^2B^2$
Dirac-Hulthén	$\frac{-B}{e^{\mu r}-1}$	$\frac{A}{e^{\mu r}-1}+1/r$	$-\sin^{-1}(\mathrm{D}B/A)$	$A^2=\mu^2+\mathrm{D}^2B^2$

- The relativistic two-point Green's function for some of these systems were obtained.

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- ………………………………

الحمد لله

و

السلام عليكم ورحمة الله وبركاته

Thank you

$$\left(i\mathrm{h}\gamma^{\mu}\partial_{\mu} - mc\right)\psi = 0 \qquad \left\{\gamma^{\mu},\gamma^{\nu}\right\} = 2\mathrm{G}^{\mu\nu}$$

$$\mathrm{h} = m = 1$$

nonrelativistic parameter

$$\mathrm{D} = \mathrm{h}/mc = 1/c \qquad \mathrm{D} \to 0$$

$$\left(i\gamma^{\mu}\partial_{\mu} - \mathrm{D}^{-1}\right)\psi = 0$$

electromagnetic coupling

$$\partial_{\mu} \to \partial_{\mu} + \frac{ie}{\mathrm{h}c} A_{\mu}$$

minimal substitution

$$\left[i\gamma^{\mu}(\partial_{\mu} + i\mathrm{D}A_{\mu}) - \mathrm{D}^{-1}\right]\psi = 0$$

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \hbar\vec{\alpha}\cdot\vec{A} + \hbar A_0 + \hbar^{-1}\beta\right)\psi \equiv \hbar^{-1}H\psi$$

$$\left[i\gamma^{\mu}(\partial_{\mu} + i\hbar A_{\mu}) - \hbar^{-1}\right]\psi = 0$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\vec{\alpha} = \gamma^0\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$H = \begin{pmatrix} +1 + \Delta^2 A_0 & -\Delta i \vec{\sigma} \cdot \vec{\nabla} + \Delta^2 \vec{\sigma} \cdot \vec{A} \\ -\Delta i \vec{\sigma} \cdot \vec{\nabla} + \Delta^2 \vec{\sigma} \cdot \vec{A} & -1 + \Delta^2 A_0 \end{pmatrix}$$

$$\vec{\sigma} \cdot \vec{A} \rightarrow \pm i \vec{\sigma} \cdot \vec{A}$$

$$H = \begin{pmatrix} 1 + \Delta^2 A_0 & -i\Delta \vec{\sigma} \cdot \vec{\nabla} + i\Delta^2 \vec{\sigma} \cdot \vec{A} \\ -i\Delta \vec{\sigma} \cdot \vec{\nabla} - i\Delta^2 \vec{\sigma} \cdot \vec{A} & -1 + \Delta^2 A_0 \end{pmatrix}$$

$$(A_0, \vec{A}) = [V(r), \hat{r} c W(r)]$$

$$\begin{pmatrix} 1 + \mathrm{D}^2 V(r) - \varepsilon & \mathrm{D}\left[\dfrac{\kappa}{r} + W(r) - \dfrac{d}{dr}\right] \\ \mathrm{D}\left[\dfrac{\kappa}{r} + W(r) + \dfrac{d}{dr}\right] & -1 + \mathrm{D}^2 V(r) - \varepsilon \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0$$

$$\kappa = \pm(j + \tfrac{1}{2}) = \pm 1, \pm 2, \ldots \qquad l = j \pm \tfrac{1}{2}$$

- Dirac-Coulomb: $V(r) = \eta / r$, $W(r) = 0$
- Dirac-Oscillator: $V(r) = 0$, $W(r) = \eta^2 r$
- S-wave Dirac-Morse:

$$\kappa = -1,\ V(r) = Ae^{-\eta r},\ W(r) = Be^{-\eta r} + \frac{1}{r}$$

- S-wave Dirac-Pöschl-Teller:

$$\kappa = -1,\ V(r) = 0,\ W(r) = A\tanh(\eta r) + \frac{1}{r}$$

$$B^2 = \eta^2 + D^2 A^2$$

$$\begin{pmatrix} 1+\mathrm{D}^2 V(r)-\varepsilon & \mathrm{D}\left[\dfrac{\kappa}{r}+W(r)-\dfrac{d}{dr}\right] \\ \mathrm{D}\left[\dfrac{\kappa}{r}+W(r)+\dfrac{d}{dr}\right] & -1+\mathrm{D}^2 V(r)-\varepsilon \end{pmatrix}\begin{pmatrix} g(r) \\ f(r) \end{pmatrix}=0$$

eliminating one spinor component in favor of the other gives a second order differential equation. This will not be Schrödinger-like (i.e., it contains first order derivatives) unless $V=0$.

$$U(\tau) = \exp\left(\tfrac{i}{2}\,\mathrm{D}\tau\sigma_2\right)$$

$$V(r) = \xi\left[W(r) + \kappa/r\right] \qquad \sin\left(\mathrm{D}\tau\right) = \pm\mathrm{D}\xi$$

$$\begin{pmatrix} C - \varepsilon + (1 \pm 1)\mathrm{D}^2\xi\left(W + \frac{\kappa}{r}\right) & \mathrm{D}\left[\mathrm{m}\xi + C\left(W + \frac{\kappa}{r}\right) - \frac{d}{dr}\right] \\ \mathrm{D}\left[\mathrm{m}\xi + C\left(W + \frac{\kappa}{r}\right) + \frac{d}{dr}\right] & -C - \varepsilon + (1\,\mathrm{ml})\mathrm{D}^2\xi\left(W + \frac{\kappa}{r}\right) \end{pmatrix} \begin{pmatrix} \phi^+(r) \\ \phi^-(r) \end{pmatrix} = 0$$

$$C = \cos(\mathrm{D}\tau) = \sqrt{1 - \mathrm{D}^2\xi^2}$$

$$\begin{pmatrix} \phi^{+} \\ \phi^{-} \end{pmatrix} = U\psi = \begin{pmatrix} \cos\frac{\mathrm{D}\eta}{2} & \sin\frac{\mathrm{D}\eta}{2} \\ -\sin\frac{\mathrm{D}\eta}{2} & \cos\frac{\mathrm{D}\eta}{2} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}$$

$$\phi^{\mathrm{m}}(r) = \frac{\mathrm{D}}{C \pm \varepsilon}\left[-\xi \pm C\left(W + \frac{\kappa}{r} \right) + \frac{d}{dr} \right] \phi^{\pm}(r)$$

$$\mathrm{D} \rightarrow 0 : \quad \varepsilon \approx 1 + \mathrm{D}^{2} E, \;\; C \approx 1 - \tfrac{1}{2}\mathrm{D}^{2}\xi^{2}$$

$$\Rightarrow \quad \phi^{-} : \; \mathrm{D}\phi^{+} \rightarrow 0$$

$$\left[-\frac{d^2}{dr^2}+\frac{C\kappa(C\kappa\pm 1)}{r^2}+2V^{\pm}(r,\varepsilon)-\frac{\varepsilon^2-1}{\mathrm{D}^2}\right]\phi^{\pm}(r)=0$$

$$V^{\pm}=\frac{1}{2}C^2W^2\ \mathrm{m}\frac{C}{2}\frac{dW}{dr}+C^2\kappa\frac{W}{r}+\xi\varepsilon W+\xi\varepsilon\frac{\kappa}{r}$$

Example

Dirac-Coulomb Problem

The Dirac-Coulomb Problem

$$V(r) = \xi\left[W(r) + \kappa/r\right]$$

$$W = 0 \rightarrow V = \xi\kappa/r \equiv Z/r$$

$$\xi = Z/\kappa$$

$$C = \sqrt{1 - \left(\mathrm{D}Z/\kappa\right)^2}$$

$$\left[-\frac{d^2}{dr^2}+\frac{\gamma(\gamma\pm 1)}{r^2}+2\frac{Z\varepsilon}{r}-\frac{\varepsilon^2-1}{\mathrm{D}^2}\right]\phi^{\pm}(r)=0$$

$$\gamma=\kappa\sqrt{1-(\mathrm{D}Z/\kappa)^2}$$

$$\left[-\frac{d^2}{dr^2}+\frac{1(1+1)}{r^2}+2\frac{Z}{r}-2E\right]\chi(r)=0$$

$$\phi^{+}:Z\rightarrow Z\varepsilon,E\rightarrow(\varepsilon^2-1)/2\mathrm{D}^2,1\rightarrow\begin{cases}\gamma\\-\gamma-1\end{cases}$$

$$\phi^{-}:Z\rightarrow Z\varepsilon,E\rightarrow(\varepsilon^2-1)/2\mathrm{D}^2,1\rightarrow\begin{cases}\gamma-1\\-\gamma\end{cases}$$

$$E_n = -Z^2 / 2(l + n + 1)^2$$

Parameter map

$$\varepsilon_n = \pm \left[1 + \left(\frac{\mathrm{DZ}}{n + |\gamma| + 1} \right)^2 \right]^{-1/2}$$

$$n = 0, 1, 2, \ldots$$

$$\chi_n(r) \sim (\rho_n r)^{1+1} e^{-\rho_n r/2} L_n^{21+1}(\rho_n r)$$

Parameter map

$$\phi_n^+(r) \sim (\mu_n r)^{\gamma+1} e^{-\mu_n r/2} L_n^{2\gamma+1}(\mu_n r)$$

$$\phi_n^-(r) \sim (\mu_{n-1} r)^{\gamma} e^{-\mu_{n-1} r/2} L_n^{2\gamma-1}(\mu_{n-1} r)$$

$$\mu_n = 2\left|Z\varepsilon_n\right| / (n + |\gamma| + 1)$$