Bosonization of a Finite Number of Non-Relativistic Fermions and Applications

Avinash Dhar

Tata Institute of Fundamental Research, Mumbai

12th Regional Conference on Mathematical Physics

National Center for Physics, Islamabad

March 31, 2006

Contents

Introduction and Motivation

- The idea of bosonization - finding a bosonic system equivalent to a given fermionic system - is almost as old as quantum mechanics itself.

Introduction and Motivation

- The idea of bosonization - finding a bosonic system equivalent to a given fermionic system - is almost as old as quantum mechanics itself.
  - Bloch - earliest observation for the existence of quantized collective bose excitations - sound waves - in a gas of fermions in 3-dimensions
  - Bohm and Pines - charge density waves - plasma oscillations - in a gas of electrons.

Introduction and Motivation

- The idea of bosonization - finding a bosonic system equivalent to a given fermionic system - is almost as old as quantum mechanics itself.
  - Bloch - earliest observation for the existence of quantized collective bose excitations - sound waves - in a gas of fermions in 3-dimensions
  - Bohm and Pines - charge density waves - plasma oscillations - in a gas of electrons.
  - Tomonaga - first important breakthrough in treating a large system of interacting fermions. In a rigorously defined simple one-dimensional model, he showed that interactions between fermions can mediate new collective bosonic d.o.f

Introduction and Motivation

- Non-relativistic fermions have a quadratic dispersion relation - Tomonaga's treatment is valid only in the low-energy approximation
- Luttinger later used a strictly linear dispersion relation. Other work - Mattis and Lieb, Haldane, ... => relativistic bosonization due to Coleman and Mandlestam
- Tomonaga-Luttinger liquid provides an important paradigm in condensed matter physics.

Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.

Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.
  - Non-critical string theory in 2-dimensions

Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.
  - Non-critical string theory in 2-dimensions
  - Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions

Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.
  - Non-critical string theory in 2-dimensions
  - Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions
  - Yang-Mills theory on a cylinder in 2-dimensions

Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.

  - Non-critical string theory in 2-dimensions
  - Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions
  - Yang-Mills theory on a cylinder in 2-dimensions

This is closely related to Tomonaga's problem

Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

$$S = \int dt \, \{\frac{1}{2}\dot{M}^2 - V(M)\}$$

Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

$$S = \int dt \, \{ \frac{1}{2} \dot{M}^2 - V(M) \}$$

M is an $N \times N$ matrix

Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

$$S = \int dt \, \{\frac{1}{2}\dot{M}^2 - V(M)\}$$

- In the $U(N)$ invariant sector, the matrix model is equivalent to a system of N non-relativistic fermions [a]
- Jevicki and Sakita [b] used this equivalence to develop a bosonization in the large-N limit - collective field theory

---

[a] Brezin, Itzykson, Parisi and Zuber, Comm. Math. Phys.59, 35, 1978

[b] Nucl.Phys.B165, 511, 1980

Introduction and Motivation

- Bosonization in terms of Wigner phase space density [a]

$$u(p,q,t) = \int dx\, e^{-ipx} \sum_{i=1}^{N} \psi_i^\dagger(q - x/2, t)\psi_i(q + x/2, t)$$

- $u(p,q,t)$ satisfies two constraints:
  - $\int \frac{dpdq}{2\pi} u(p,q,t) = N$
  - $u * u = u$

[a] Dhar, Mandal and Wadia, hep-th/9204028; 9207011; 9309028

Introduction and Motivation

- Bosonization in terms of Wigner phase space density [a]

$$u(p,q,t) = \int dx\; e^{-ipx} \sum_{i=1}^{N} \psi_i^\dagger(q - x/2, t)\psi_i(q + x/2, t)$$

- $u(p,q,t)$ satisfies two constraints:
  - $\int \frac{dpdq}{2\pi} u(p,q,t) = N$
  - $u * u = u$

Many more variables than are necessary

[a] Dhar, Mandal and Wadia, hep-th/9204028; 9207011; 9309028

Exact Bosonization

The Setup:

- each can occupy a state in an infinite-dimensional Hilbert space $\mathcal{H}_f$
- there is a countable basis of $\mathcal{H}_f : \{|m\rangle, m = 0, 1, \cdots, \infty\}$
- creation and annihilation operators $\psi^\dagger_m$, ψ_m create and destroy particles in the state $|m\rangle, \quad \{\psi_m, \psi^\dagger_n\} = \delta_{mn}$
- total number of fermions is fixed:

$$\sum_n \psi^\dagger_n \psi_n = N$$

Exact Bosonization

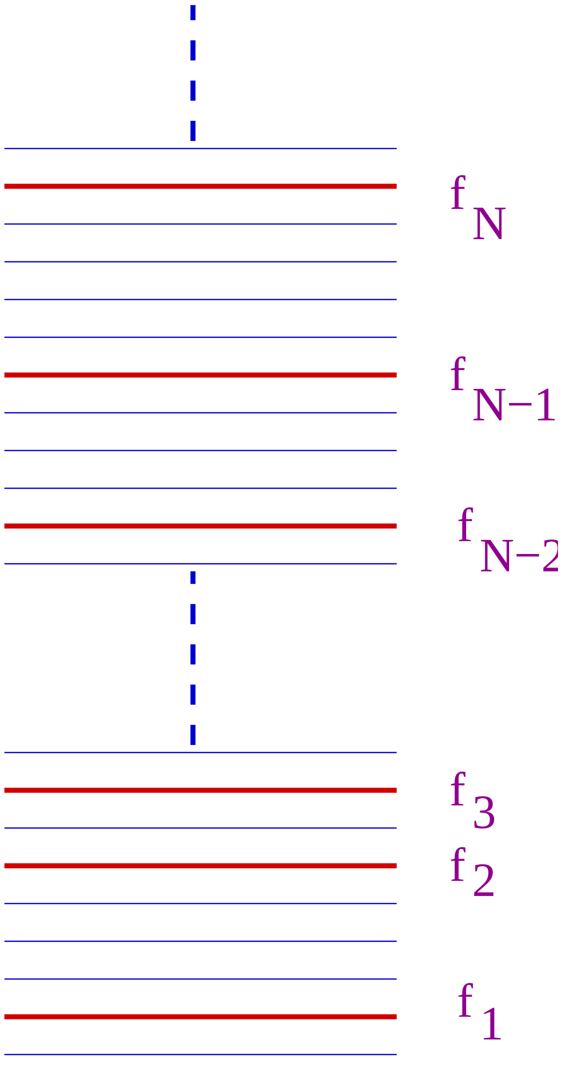
- The N-fermion states are given by (linear combinations of)

$$|f_1, \cdots, f_N\rangle = \psi^\dagger_{f_N} \cdots \psi^\dagger_{f_2} \psi^\dagger_{f_1} |0\rangle_F,$$

  - $|0\rangle_F$ is Fock vacuum
  - f_k are ordered $0 \leq f_1 < f_2 < \cdots < f_N$

- Repeated applications of the bilinear $\psi^\dagger_m \psi_n$ gives any desired state

Exact Bosonization



Exact Bosonization

Bosonization: [a]

- Introduce the bosonic operators

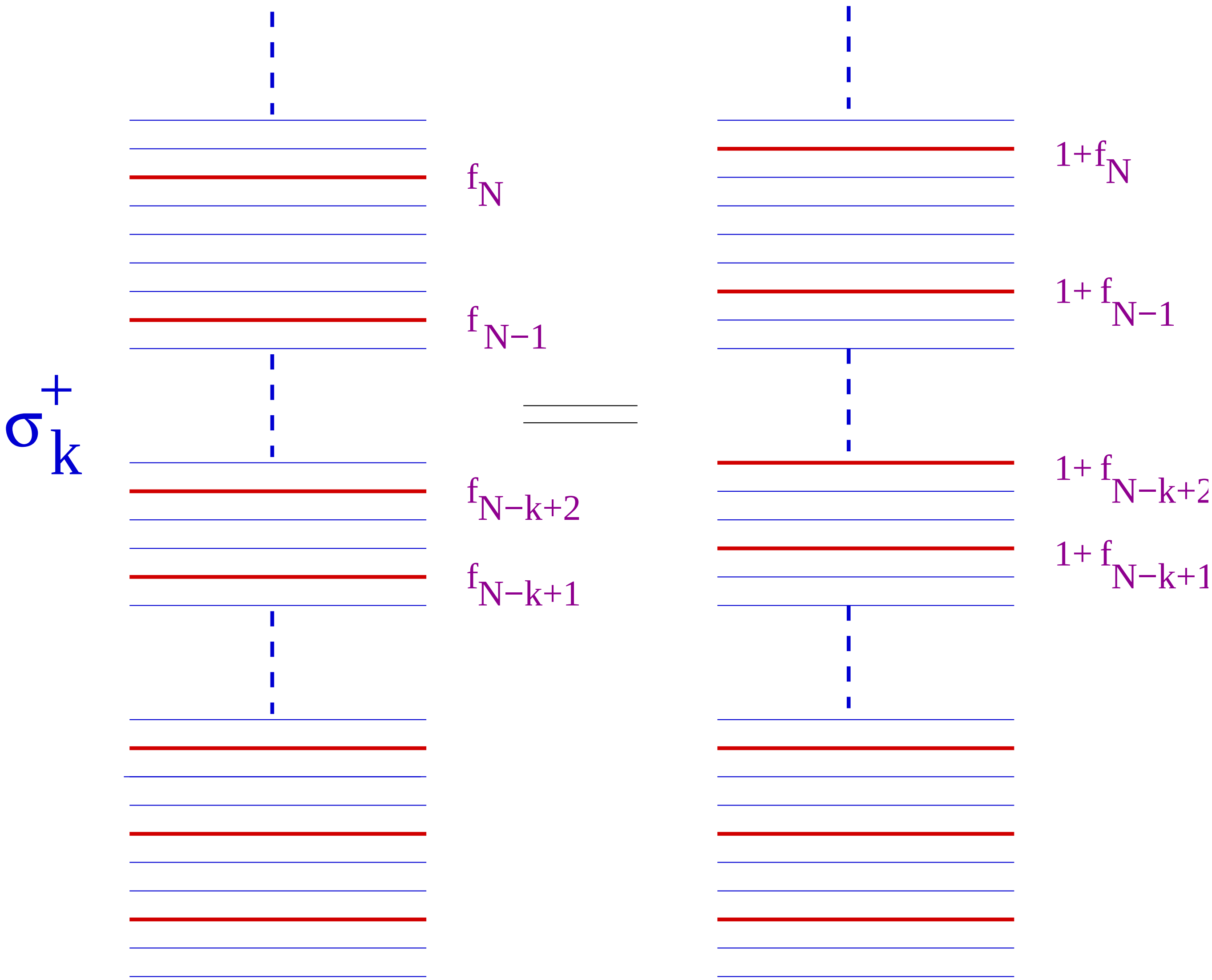
$$\sigma_k, \; k = 1, 2, \cdots, N$$

- and their conjugates

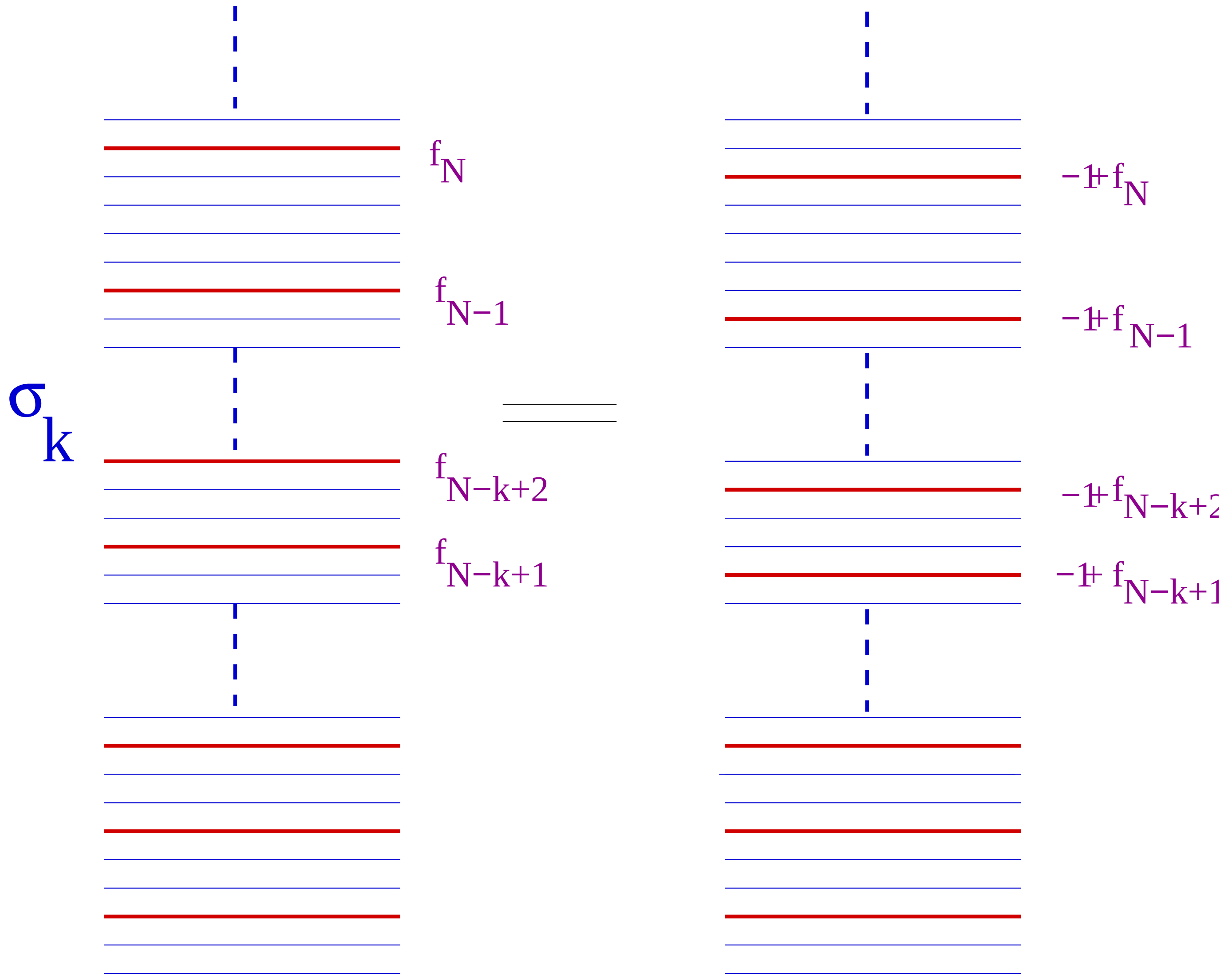
$$\sigma_k^\dagger, \; k = 1, 2, \cdots, N$$

[a] Dhar, Mandal and Suryanarayana, hep-th/0509164

Exact Bosonization



Exact Bosonization

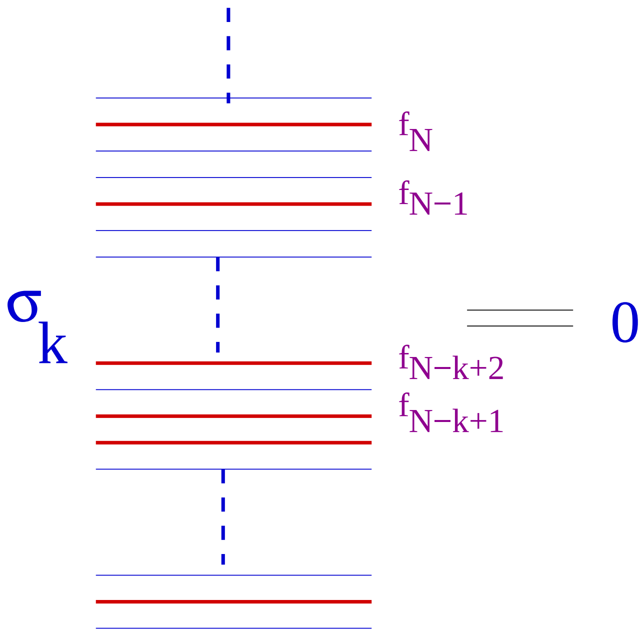


Exact Bosonization

- By definition:
  - $\sigma_k \sigma_k^\dagger = 1$
  - $\sigma_k^\dagger \sigma_k = 1$, if σ_k does not annihilate the state

Exact Bosonization

- By definition:
  - $\sigma_k \sigma_k^\dagger = 1$
  - $\sigma_k^\dagger \sigma_k = 1$, if σ_k does not annihilate the state



Exact Bosonization

- By definition:
  - $\sigma_k \sigma_k^\dagger = 1$
  - $\sigma_k^\dagger \sigma_k = 1$, if σ_k does not annihilate the state
- For $k \neq l$, $[\sigma_k, \sigma_l^\dagger] = 0$

Exact Bosonization

- Introduce creation (annihilation) operators $a_k^\dagger$ (a_k) which satisfy the standard commutation relations

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad k, l = 1, \cdots, N$$

- The states of the bosonic system are given by (a linear combination of)

$$|r_1, \cdots, r_N\rangle = \frac{(a_1^\dagger)^{r_1} \cdots (a_N^\dagger)^{r_N}}{\sqrt{r_1! \cdots r_N!}} |0\rangle$$

Exact Bosonization

- Now, make the following identifications

$$\sigma_k = \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k; \qquad \sigma_k^\dagger = a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}}$$

- together with the map [a]

$$r_N = f_1; \quad r_k = f_{N-k+1} - f_{N-k} - 1, \quad k = 1,\, 2,\, \cdots\, N-1$$

- For the Fermi vacuum, $f_{k+1} = f_k + 1$ and so $r_k = 0$ for all k => Fermi vacuum = Bose vacuum

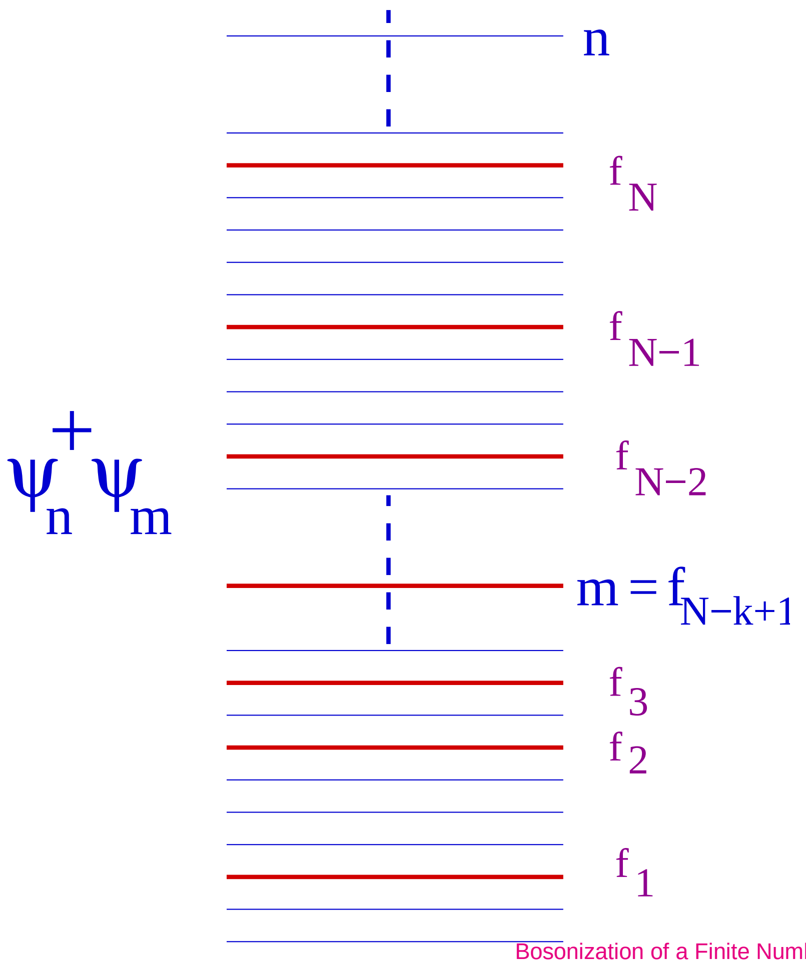
[a] Suryanarayana, hep-th/0411145

Exact Bosonization

- The σ_k, $k = 1, 2, \cdots, N$ are necessary and sufficient
  - Any bilinear $\psi_n^\dagger \psi_m$ can be built out of σ_k's

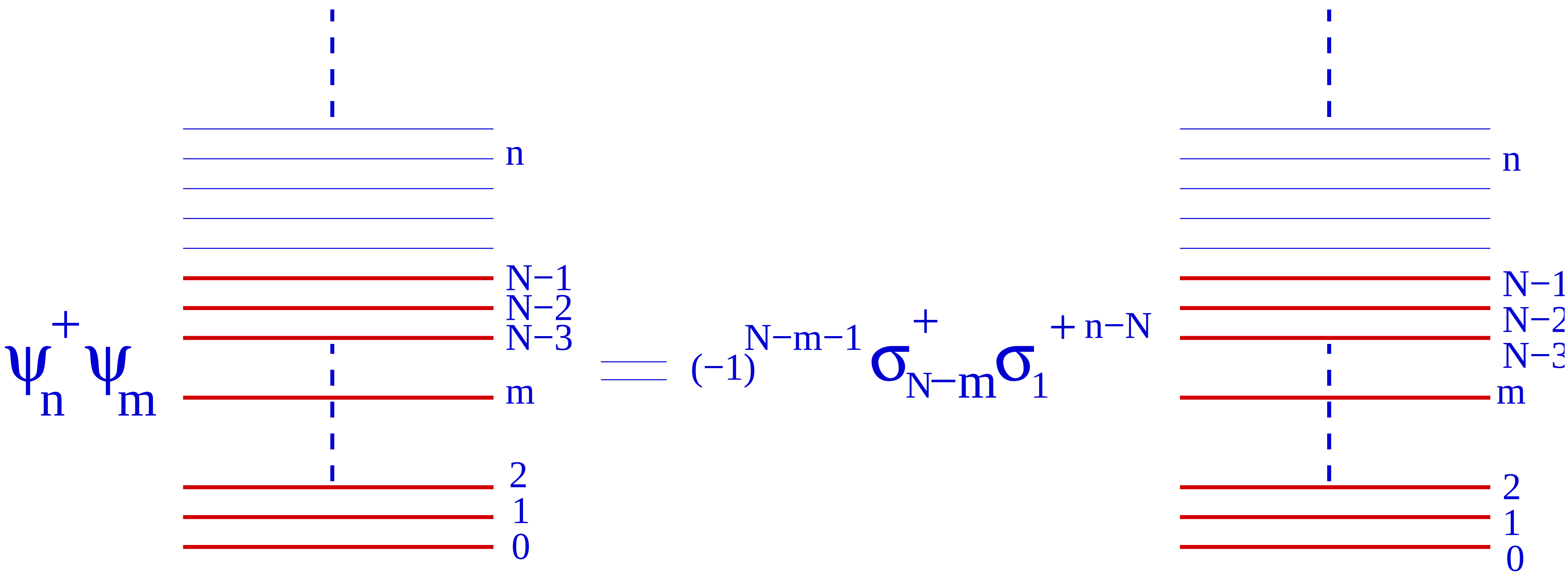
Exact Bosonization

- The σ_k, $k = 1, 2, \cdots, N$ are necessary and sufficient
  - Any bilinear $\psi_n^\dagger \psi_m$ can be built out of σ_k's



Exact Bosonization

- The σ_k, $k = 1, 2, \cdots, N$ are necessary and sufficient
  - Any bilinear $\psi_n^\dagger \psi_m$ can be built out of σ_k's



Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!

Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!
- There is no natural “space” in the bosonic theory - in the examples we will discuss, a spatial direction will emerge in the low-energy large-N limit.

Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!
- There is no natural "space" in the bosonic theory - in the examples we will discuss, a spatial direction will emerge in the low-energy large-N limit.
  - In applications involving matrix quantum mechanics, our bosonization can be considered to be an exact solution of the matrix problem in the singlet sector

Exact Bosonization

- The non-interacting fermionic Hamiltonian:

$$H = \sum_n \mathcal{E}(n)\psi_n^\dagger \psi_n$$

Exact Bosonization

- The non-interacting fermionic Hamiltonian:

$$H = \sum_n \mathcal{E}(n)\psi_n^\dagger \psi_n$$

- The bosonized Hamiltonian:

$$H = \sum_{k=1}^{N} \mathcal{E}(\hat{n}_k), \quad \hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$$

Exact Bosonization

- The non-interacting fermionic Hamiltonian:

$$H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n$$

- The bosonized Hamiltonian:

$$H = \sum_{k=1}^{N} \mathcal{E}(\hat{n}_k), \quad \hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$$

- What about fermion interactions? These can also be included since the generic bilinear $\psi_n^\dagger \psi_m$ has a bosonized expression

Half-BPS states and LLM geometries

- Space-time, gravity lagrangians and gravitons - low-energy emergent properties of an underlying microscopic dynamics
- String theory is a consistent theory of quantum gravity => we should be able to test these ideas
- AdS/CFT correspondence => a precise setting in which to explore these ideas

Half-BPS states and LLM geometries

- Space-time, gravity lagrangians and gravitons - low-energy emergent properties of an underlying microscopic dynamics
- String theory is a consistent theory of quantum gravity => we should be able to test these ideas
- AdS/CFT correspondence => a precise setting in which to explore these ideas
  - Classic example of dual pair - $\mathcal{N} = 4$ SYM and string theory on AdS$_5 \times$S^5. No hint of 10-d space-time or gravitons in the SYM theory!
  - This duality => weakly coupled low-energy type IIB gravity on AdS$_5 \times$S^5 and strongly coupled $\mathcal{N} = 4$ SYM theory in the large-N limit have exactly the same physical content

Half-BPS states and LLM geometries

- LLM work [a] - a small new window of opportunity.
- Limited to half-BPS sector, but hopefully has some wider lessons
  - SYM - half-BPS states are described by a holomorphic sector of quantum mechanics of an $N \times N$ complex matrix Z in a harmonic potential
  - This system can be shown [b] to be equivalent to the quantum mechanics of an $N \times N$ hermitian matrix Z in a harmonic potential

---

[a] Lin, Lunin and Maldacena, hep-th/0409174

[b] Takayama and Tsuchiya, hep-th/0507070

Half-BPS states and LLM geometries

- Gauge invariance => physical observables on boundary are $U(N)$-invariant traces:

$$\mathrm{tr} Z^k, \quad k = 1, 2, \cdots, N$$

- Physical states <=> operators

$$(\mathrm{tr} Z^{k_1})^{l_1} (\mathrm{tr} Z^{k_2})^{l_2} \cdots$$

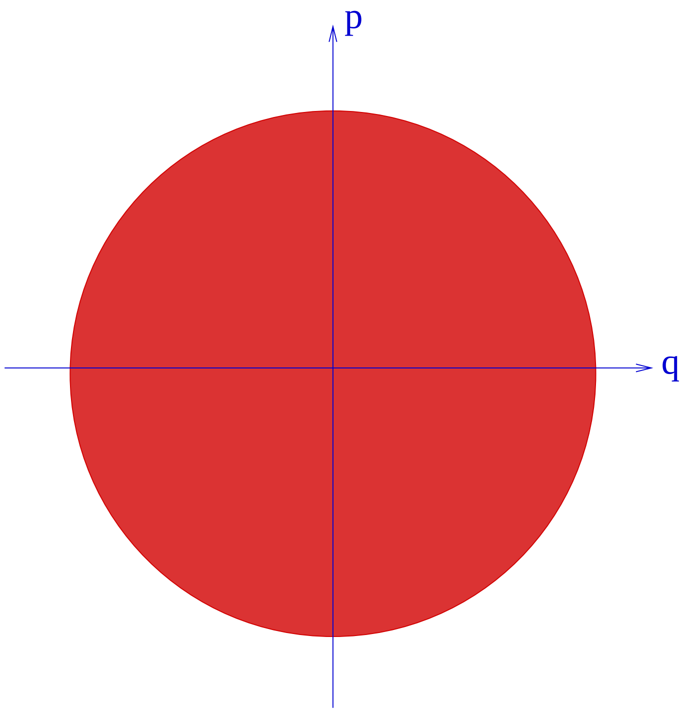
- Total number of Z's is a conserved RR charge $Q = \sum_i k_i l_i$. BPS condition => $E = Q$

Half-BPS states and LLM geometries

- At large N there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space

Half-BPS states and LLM geometries

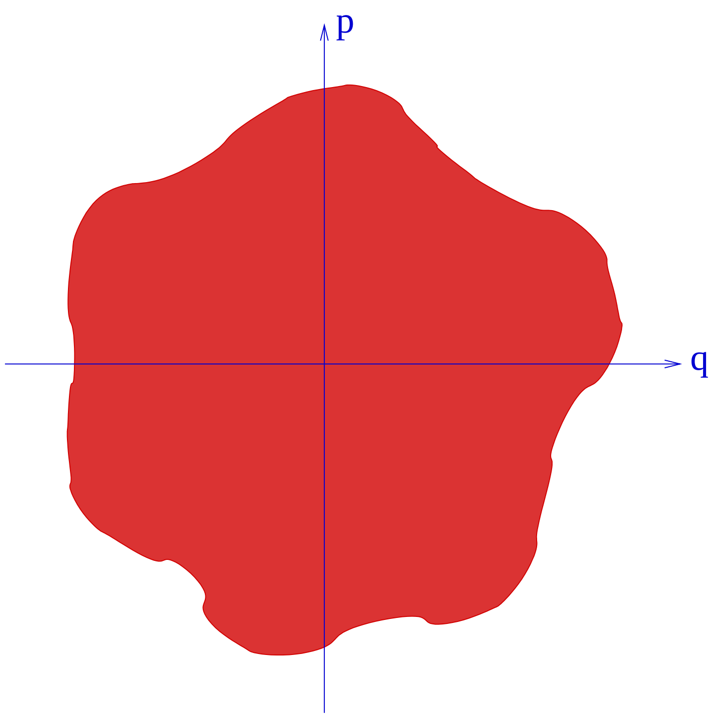
- At large N there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space



ground state distribution

Half-BPS states and LLM geometries

- At large N there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space



small fluctuations around the ground state

Half-BPS states and LLM geometries

- By explicitly solving equations of type IIB gravity, LLM showed that there is a similar structure in the classical geometries in the half-BPS sector!
- LLM solutions - two of the space coordinates are identified with the phase space of a single fermion => noncommutativity in two space directions in the semicalssical description [a]
- Small fluctuations around AdS space, i.e low-energy graviton excitations [b] [c] $\equiv$ low-energy fluctuations of the fermi vacuum [d]

[a] Mandal, hep-th/0502104

[b] Grant, Maoz, Marsano, Papadodimas and Rychkov, hep-th/0505079

[c] Maoz and Rychkov, hep-th/0508059

[d] Dhar, hep-th/0505084

Half-BPS states and LLM geometries

- Motivation for our work [a] - on the CFT side the half-BPS system can be quantized exactly in terms of our bosons => window of opportunity to learn about aspects of quantum gravity.
- At finite N, only the low-energy excitations on the boundary can be identified with low-energy ($<< N$) gravitons in the bulk
- The single-particle graviton excitations are related to our bosons. On the boundary, these states are:

$$\beta_m^\dagger|0\rangle = \sum_{n=1}^{m}(-1)^{n-1}\sqrt{\frac{(N+m-n)!}{2^m(N-n)!}}{\sigma_1^\dagger}^{m-n}\sigma_n^\dagger|0\rangle$$

[a] Dhar, Mandal and Smedback, hep-th/0512312

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- On the boundary:

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- On the boundary:
  - at low energies, perturbation theory is good and reproduces supergravity answers; there is an effective cubic hamiltonian

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- On the boundary:
  - at low energies, perturbation theory is good and reproduces supergravity answers; there is an effective cubic hamiltonian
  - perturbation theory breaks down for β's with energy of order $\sqrt{N}$

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1}\beta^\dagger_{k_2}\cdots >$
- On the boundary:
  - at low energies, perturbation theory is good and reproduces supergravity answers; there is an effective cubic hamiltonian
  - perturbation theory breaks down for β's with energy of order $\sqrt{N}$
  - At energies of order N, the β interactions grow exponentially with N

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- In the bulk:

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- In the bulk:
  - gravitons with energies larger than $\sqrt{N}$ have a size smaller than 10-dim planck scale

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- In the bulk:
  - gravitons with energies larger than $\sqrt{N}$ have a size smaller than 10-dim planck scale
  - nonlocal solitonic excitations with energy of order N - giant gravitons [a]

---

[a] McGreevy, Susskind and Toumbas, hep-th/0003075

Half-BPS states and LLM geometries

- One can compute exactly the correlation functions $< \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots >$
- In the bulk:
  - gravitons with energies larger than $\sqrt{N}$ have a size smaller than 10-dim planck scale
  - nonlocal solitonic excitations with energy of order N - giant gravitons [a]
  - The size of giant gravitons is larger than 10-dim planck scale for energies larger than $\sqrt{N}$

[a] McGreevy, Susskind and Toumbas, hep-th/0003075

Half-BPS states and LLM geometries

- On the boundary, single-particle giant graviton states map to linear combinations of multi-graviton states [a]. Example:

$$|\text{giant graviton of energy } 2\rangle = ({\beta_1^\dagger}^2 - \beta_2^\dagger)|0\rangle$$

[a] Balasubramanian, Berkooz, Naqvi and Strassler, hep-th/0107119

Half-BPS states and LLM geometries

- On the boundary, single-particle giant graviton states map to linear combinations of multi-graviton states [a]. Example:

$$|\text{giant graviton of energy } 2\rangle = ({\beta_1^\dagger}^2 - \beta_2^\dagger)|0\rangle = a_2^\dagger|0\rangle$$

[a] Balasubramanian, Berkooz, Naqvi and Strassler, hep-th/0107119

Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a_k^\dagger |0\rangle$$

Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a_k^\dagger|0\rangle$$

- Hamiltonian: $H_F = \sum_n \mathcal{E}(n)\psi_n^\dagger\psi_n => H_B = \sum_{k=1}^N k a_k^\dagger a_k$

Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a_k^\dagger|0\rangle$$

- Hamiltonian: $H_F = \sum_n \mathcal{E}(n)\psi_n^\dagger\psi_n \Rightarrow H_B = \sum_{k=1}^N k a_k^\dagger a_k$

- Discrete space?

$$\phi(\theta_n) = \sum_{k=1}^N (e^{ik\theta_n} a_k + e^{-ik\theta_n} a_k^\dagger), \quad \theta_n = \frac{2\pi n}{N}$$

Half-BPS states and LLM geometries

Summary (half-BPS sector):

- low-energy fluctuations of the metric around AdS are adequately described by gravitons
- at moderately high energies of order $\sqrt{N}$, perturbative gravity breaks down; one must now sum to all orders in $1/N$ to get correct answers
- at very high energies of order N, gravitons cease to provide a meaningful description; instead we must now use a new set of d.o.f., namely the giant gravitons, which are weakly coupled at high energies

Free fermions on a circle

- We will mainly discuss [a] the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.
- The free hamiltonian:

$$H = -\frac{\hbar^2}{2m}\int_0^L dx\ \chi^\dagger(x)\partial_x^2\chi(x)$$

[a] Dhar and Mandal, hep-th/0603154

Free fermions on a circle

- We will mainly discuss [a] the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

- Hamiltonian in terms of fourier modes:

$$H = \omega\hbar \sum_{n=-\infty}^{\infty} n^2 \chi_n^\dagger \chi_n, \qquad \omega \equiv \frac{2\pi^2\hbar}{mL^2}$$

[a] Dhar and Mandal, hep-th/0603154

Free fermions on a circle

- We will mainly discuss [a] the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

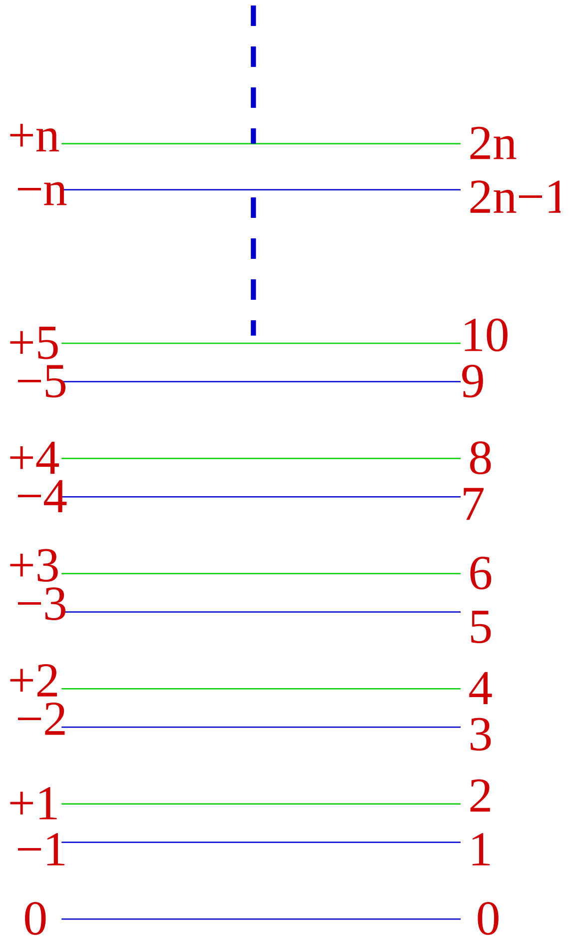
- Hamiltonian in terms of fourier modes:

$$H = \omega\hbar \sum_{n=-\infty}^{\infty} n^2 \chi_n^\dagger \chi_n, \qquad \omega \equiv \frac{2\pi^2\hbar}{mL^2}$$

- To apply our bosonization rules, need to introduce an ordering in the spectrum. For example, replace $n^2 \longrightarrow (n+\epsilon)^2$

[a] Dhar and Mandal, hep-th/0603154

Free fermions on a circle



Free fermions on a circle

- Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$

Free fermions on a circle

- Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$
- Fermionic hamiltonian:

$$H = \omega\hbar \sum_{n=1}^{\infty} \left(\frac{n + e(n)}{2} \right)^2 \psi_n^\dagger \psi_n$$

Free fermions on a circle

- Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$
- Fermionic hamiltonian:

$$H = \omega\hbar \sum_{n=1}^{\infty} \left(\frac{n + e(n)}{2} \right)^2 \psi_n^\dagger \psi_n$$

- Bosonized hamiltonian:

$$H = \omega\hbar \sum_{k=1}^{N} \left(\frac{\hat{n}_k + e(\hat{n}_k)}{2} \right)^2$$

where $\hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$

Free fermions on a circle

- Large-N low energy limit: $H = H_F + H_0 + H_1$

Free fermions on a circle

- Large-N low energy limit: $H = H_F + H_0 + H_1$

$$H_0 = \frac{\hbar\omega N}{2}\left(\sum_{k=1}^{N} k\; a_k^\dagger a_k + \hat{\nu}\right)$$

  - $\hat{\nu} = N_- - N_{-F} = \sum_{k=1}^{N}(e(\hat{n}_k) - e(N-k))$ is the number of excess fermions in negative momentum states over and above the number in fermi vacuum
  - H_1 is order one on excited states whose energy is low compared to N

Free fermions on a circle

- The massless collective boson:

Free fermions on a circle

- The massless collective boson:
  - The partition function

$$Z_N = \sum e^{-\beta H_0}$$

Free fermions on a circle

- The massless collective boson:
  - The partition function

$$Z_N = \sum e^{-\beta H_0}$$

  - In the limit $N \to \infty$, the partition function turns out to be

$$Z_\infty = \sum_{\nu=-\infty}^{+\infty} q^{\nu^2} \left[\prod_{n=1}^{\infty} (1 - q^n)^{-1} \right]^2 ; \quad q = e^{-\hbar\omega N\beta}$$

Free fermions on a circle

- States ($\nu = 0$):

Free fermions on a circle

- States ($\nu = 0$):
  - states at level l have energy $E_{0l} = \hbar\omega N l$ - these include multiparticle states of both chiralities

Free fermions on a circle

- States ($\nu = 0$):
  - states at level l have energy $E_{0l} = \hbar\omega N l$ - these include multiparticle states of both chiralities
  - example, $l = 2$:

$$(\sigma_1^\dagger)^4|0\rangle,\ \sigma_1^\dagger\sigma_3^\dagger|0\rangle,\ (\sigma_{\bar{1}}^\dagger)^2\sigma_{\bar{2}}^\dagger|0\rangle,\ \sigma_{\bar{4}}^\dagger|0\rangle,\ (\sigma_{\bar{2}}^\dagger)^2|0\rangle$$

  - first two have momentum of opposite sign to the next two; the last state has zero momentum
  - first four states: two single-particle and two 2-particle states of each chirality; the last state is non-chiral 2-particle state

Free fermions on a circle

- $1/N$ corrections:

Free fermions on a circle

- $1/N$ corrections:
  - linear combinations exist in which expectation value of H_1 vanishes:

$$\frac{1}{\sqrt{2}}[(\sigma_1^\dagger)^4 \pm \sigma_1^\dagger \sigma_3^\dagger]|0\rangle; \quad \frac{1}{\sqrt{2}}[(\sigma_1^\dagger)^2 \sigma_2^\dagger \pm \sigma_4^\dagger]|0\rangle$$

Free fermions on a circle

- $1/N$ corrections:
  - linear combinations exist in which expectation value of H_1 vanishes:

$$\frac{1}{\sqrt{2}}[(\sigma_1^\dagger)^4 \pm \sigma_1^\dagger \sigma_3^\dagger]|0\rangle; \qquad \frac{1}{\sqrt{2}}[(\sigma_1^\dagger)^2 \sigma_2^\dagger \pm \sigma_4^\dagger]|0\rangle$$

  - such linear combinations exist at all levels; at each level there is one linear combination which is identical to an appropriate mode of the fermion density!

$$\frac{1}{\sqrt{l}} \sum_n \psi_{n+2l}^\dagger \psi_n |F_0\rangle \equiv \rho_l^\dagger |0\rangle = \frac{1}{\sqrt{l}} \sum_k^{[2l]} (\sigma_1^\dagger)^{2l-k} \sigma_k^\dagger |0\rangle$$

Free fermions on a circle

- Cubic interaction:

Free fermions on a circle

- Cubic interaction:

$$H_1 \rho_l^\dagger |0\rangle = \hbar\omega \sum_{m=1}^{l-1} c_m^l \rho_m^\dagger \rho_{(l-m)}^\dagger |0\rangle, \quad l \leq \frac{N+1}{2}$$

Free fermions on a circle

- Cubic interaction:

$$H_1 \rho_l^\dagger |0\rangle = \hbar\omega \sum_{m=1}^{l-1} c_m^l \rho_m^\dagger \rho_{(l-m)}^\dagger |0\rangle, \quad l \leq \frac{N+1}{2}$$

  - the coefficient:

$$c_m^l = \sqrt{lm(l-m)}$$

Free fermions on a circle

- Beyond low-energy perturbation theory:

Free fermions on a circle

- Beyond low-energy perturbation theory:
  - states created by modes of fermion density have an extra term at high energies:

$$\tilde{\rho}_l^\dagger|0\rangle = \rho_l^\dagger|0\rangle + \frac{1}{\sqrt{l}}\sum_{n=1}^{l}\psi^\dagger_{2(l-n)}\psi_{2n-1}|0\rangle$$

Free fermions on a circle

- Beyond low-energy perturbation theory:
  - states created by modes of fermion density have an extra term at high energies:

$$\tilde{\rho}_l^\dagger|0\rangle = \rho_l^\dagger|0\rangle + \frac{1}{\sqrt{l}}\sum_{n=1}^{l}\psi^\dagger_{2(l-n)}\psi_{2n-1}|0\rangle$$

  - last term cannot be ignored at high energies; it is in fact a $\nu = -1$ state!

Free fermions on a circle

- $$\tilde{\rho}^\dagger_m \tilde{\rho}^\dagger_{(\frac{N+3}{2}-m)}|0\rangle = \rho^\dagger_m \rho^\dagger_{(\frac{N+3}{2}-m)}|0\rangle + \frac{1}{\sqrt{m(\frac{N+3}{2}-m)}} \sigma^\dagger_1 \sigma^\dagger_{N-1}|0\rangle$$

- last term will contribute in

$$\langle 0|\tilde{\rho}^\dagger_{+\frac{N+3}{2}} \tilde{\rho}^\dagger_{+m} \tilde{\rho}^\dagger_{+(\frac{N+3}{2}-m)}|0\rangle$$

Free fermions on a circle

- Exact partition function for finite N (H_0 part only):

$$Z_N = \sum_{\nu=-\frac{N-1}{2}}^{+\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{\frac{N+1}{2}-\nu} (1-q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1-q^n)^{-1}$$

Free fermions on a circle

- Exact partition function for finite N (H_0 part only):

$$Z_N = \sum_{\nu=-\frac{N-1}{2}}^{+\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{\frac{N+1}{2}-\nu} (1-q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1-q^n)^{-1}$$

- For large-N:

$$Z_N = (1+2q-q^{\frac{N+1}{2}})(1-q^{\frac{N+1}{2}})\left[\prod_{n=1}^{\infty}(1-q^n)^{-1}\right]^2$$

Free fermions on a circle

- Exact partition function for finite N (H_0 part only):

$$Z_N = \sum_{\nu=-\frac{N-1}{2}}^{+\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{\frac{N+1}{2}-\nu} (1-q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1-q^n)^{-1}$$

- For large-N:

$$Z_N = (1 + 2q - q^{\frac{N+1}{2}})(1 - q^{\frac{N+1}{2}}) \left[\prod_{n=1}^{\infty} (1-q^n)^{-1} \right]^2$$

O(e^{-N})

  - nonperturbative effects in 2-d YM
  - black-hole counting and baby universes

Free fermions on a circle

Summary:

- Tomonaga's problem has an exact solution in terms of our bosons. Low-energy local cubic collective field theory can be derived; the collective field is a linear combination of multi-particle states of our bosons, like the graviton in the LLM case

- our bosonization goes beyond this low-energy local limit, but then there is no natural local space-time field theory interpretation

- density-density interactions, as in a system of electrons with Coulomb interactions, can be incorporated - easy at low-energies, but requires more work at high energies

SUMMARY

- We have developed a simple and exact bosonization of a finite number of non-relativistic fermions; we discussed here applications to concrete problems in different areas of physics
- Our bosonization trades finiteness of the number of fermions for finite dimensionality of the single-particle boson Hilbert space
- the bosonized theory is inherently grainy; in the specific applications we discussed, a local space-time field theory emerges only in the large-N and low-energy limit
- Bosonization of finite number of fermions in higher dimensions?