Some New Casimir Energy Calculations
(with H. Ahmedov)

- Closed surfaces, JMP, 44, 5487 (2003)
- Triangular cross section, JMP, 45, 965 (2004)
- Pyramidal cavity, JMP, 46, 022303 (2005)
- Conical Cavity, JMP, 46, 022304 (2005)

Motivations

- To have as much as possible examples in hand; then hope to learn "what governs the sign of Casimir energy?"
- In the light of the rapid advancements in Nanotechnology we hope that some of the results may be tested by experiment
- For a cavity of typical size of a, the energy for massless scalar field

$$|E| \simeq (10^{-2} \to 10^{-1})/a \ \text{cm}^{-1}$$

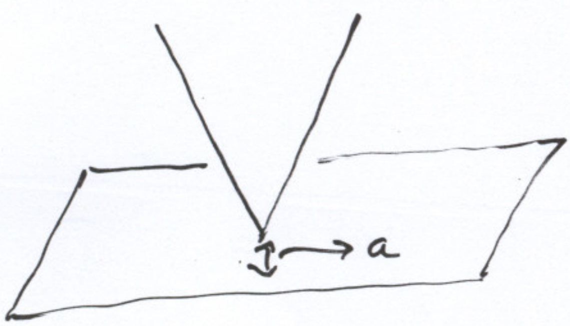
(For e-m field $|E|_{em} \simeq 2|E|$)

for nanometer size

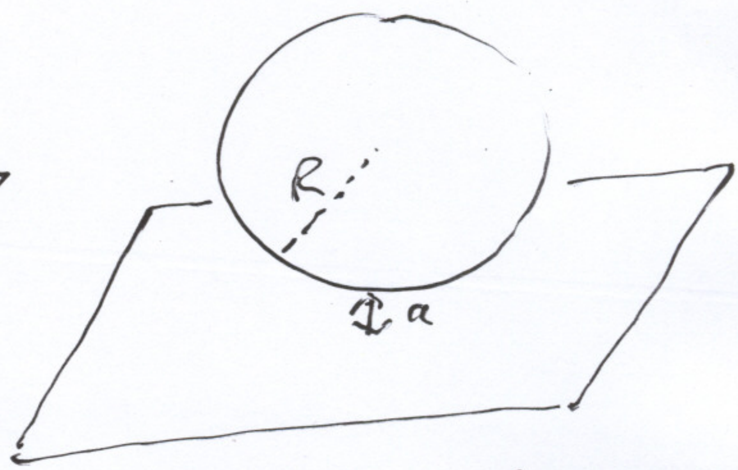
$$|E| \simeq (10 \to 100)\,\text{eV} \qquad 1\,\text{eV} \cong 0.5 \times 10^{5}\,\text{cm}^{-1}$$

Experimental Status

- All measurements are done for 2 body geometries:



Cone before plane



Ball before plane
$(R \gg a)$

Original parallel plates
G. Bressi et al, Phys. Rev. Lett. 88, 041804 (2002)

Area of plates $1.2 \times 1.2\ \mathrm{mm}^2$

$a \simeq 0.4\ \mu m$

The Force $F = -\frac{\partial E}{\partial a} = -\frac{\pi^2 \hbar c}{240} \frac{1}{a^4}$ per unit area of plates

is measured (by piezoelectricity) current.

(% 15 precision.)

$$F = -\frac{1.3 \times 10^{-27}\ N\, m^2}{a^4}$$

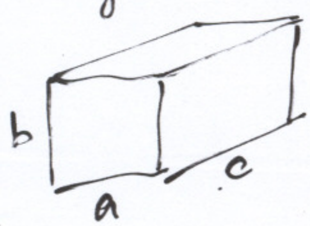
All previous experiments are essentially qualitative

• No single Cavity experiments has been performed yet.

• Previously calculated single cavity energies:

(i) Spherical Cavity $E_{Ball} \simeq \frac{0.045}{R} > 0$

(ii) Rectangular Cavities (of several dimensions of spaces)



Sign of the energy depends on the ratios of a, b, c and on the dimension of space!

In 3-space dimension the energy of cube of edges a is

$$E_{cube} \simeq -\frac{0.014}{a} < 0 \; !$$

• All are for e-m field.

• For massive fields $E_m \simeq \underset{\substack{\downarrow \\ \text{Massless field} \\ \text{Energy}}}{E_0} \, e^{-mc}$ (in $\hbar = c = 1$ unit)

(i) For electron $mc \simeq 2500$ ⇒ For massive fields it is practically zero!

⇓

(ii) For neutrino field there is no wall !! May be important ~~for~~ in cosmology..

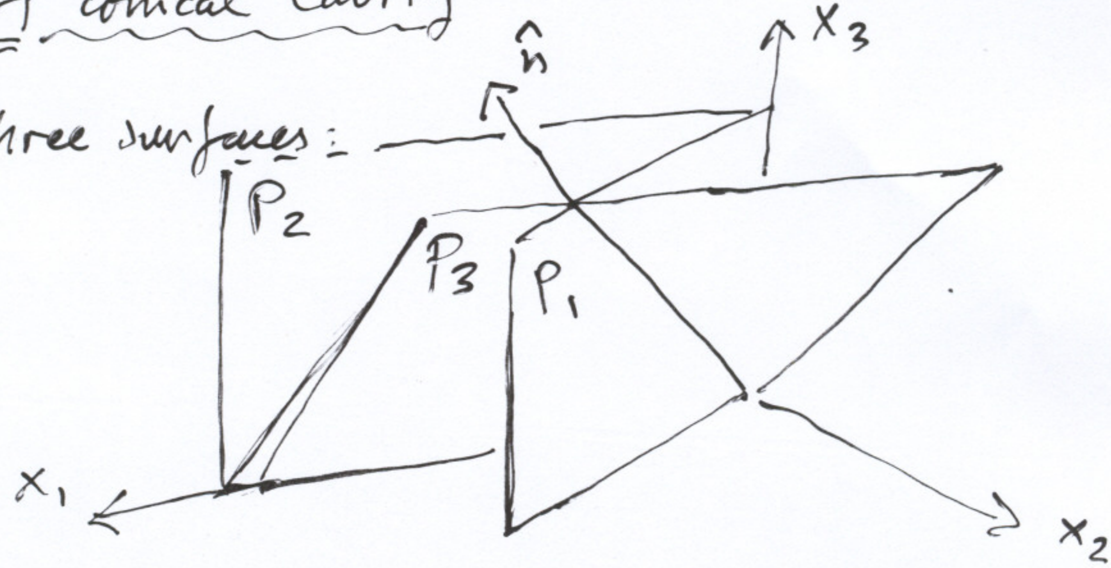
Our Results

- All are for massless scalar field
- For em field roughly 2 times the energy for massless scalar field is obtained!
- The cavities are rather special, for these shapes one can keep tracks of the images in the process of the calculations of the required Green functions.

List of the results:

(i) $\exists$ conical cavity

Three surfaces:



$P_1: \; x_1 = x_2$

$P_2: \; x_2 = 0$

$P_3: \; x_2 = x_3$

$\hat{n}$: intersection of planes P_1 and P_3

Boundary conditions (1) $K(x,y)\big|_{\vec{x}\in P_2} = 0$

(2) $K(x,y)\big|_{\vec{x}\in P_1} = K(x,y)\big|_{g\vec{x}\in P_3}$

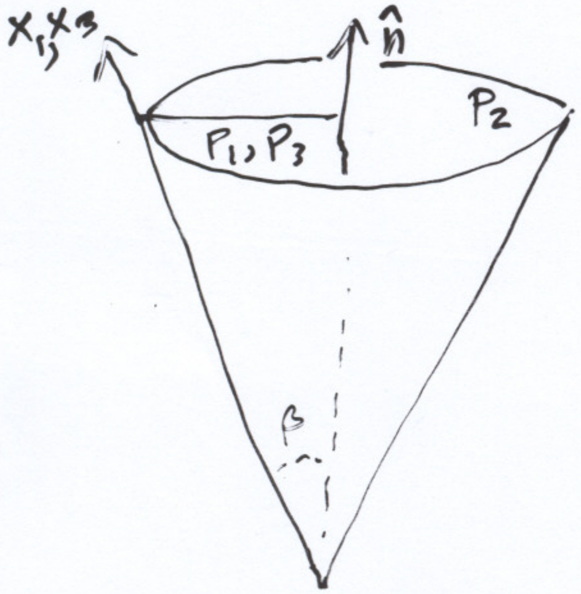
g = rotation matrix around $\hat{n}$ by angle $2\pi/3$

by condition (2)

and by Drichlet boundary condition on planes

$$P_4: \ x_1 = a \ , \ P_5: \ x_3 = a$$

we arrive at the conical cavity



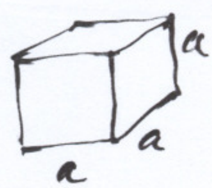
$\beta = \arcsin \frac{1}{3} \simeq 39^\circ$

Energy in it is

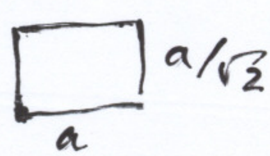
$$E_{con} \simeq \frac{1}{3} E_1 - \frac{1}{2} E_2 - \frac{2+\sqrt{2}}{4} E_3 \simeq \frac{0.035}{a} > 0$$

where

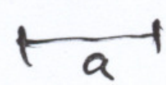
$E_1 \simeq -\frac{0.015}{a}$ energy for cube



$E_2 \simeq \frac{0.045}{a}$ " for rectangle



$E_1 \simeq -\frac{0.131}{a}$ " for length

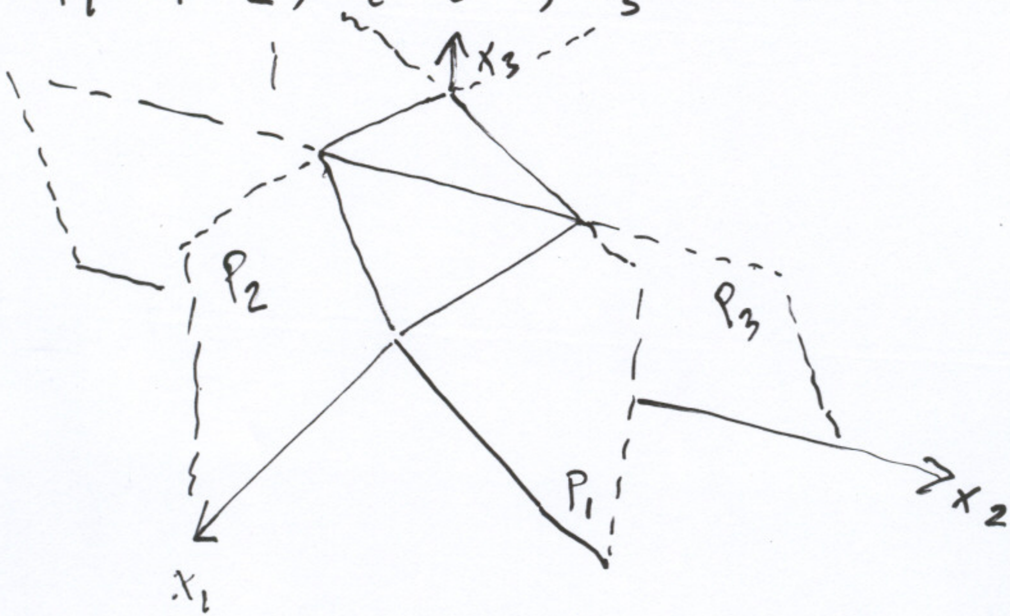


[Notice the alteration of sign by dimension of the geometry.

(ii) A Pyramidal Cavity

Region defined by Drichlet Boundary conditions on Planes:

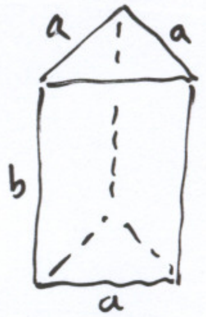
$P_1: x_1 = x_2$, $P_2: x_2 = 0$, $P_3: x_1 = x_3$ and $P_4: x_3 = a$



$$E_{pyra} \simeq \frac{0.069}{a} > 0$$

(iii) A prism with triangular cross section

Cross section is equilateral triangle of edge a



$b > a$ $$E_{tri} \simeq -\frac{0.053}{a} + \frac{(0.029)\,b}{a^2}$$

$$E_{tri} \simeq \frac{1}{2}\left(-\frac{0.013}{b} + \frac{(0.011)\,a}{b^2} + \frac{0.093}{a} - \frac{(0.048)\,b}{a^2}\right)$$

$a > b > \frac{a}{\sqrt{2}} \simeq 0.7a$

$b < a/\sqrt{2} \simeq 0.7a$

$$E_{tri} \simeq -\frac{0.039}{b} + \frac{(0.014)\,a}{b^2}$$

The Casimir forces in vertical direction F_b and in horizontal directions F_a on the surfaces are:

For $b > a/\sqrt{2}$ $\quad F_b < 0$, $F_a > 0$

For $b < a/\sqrt{2}$ $\quad F_b > 0$, $F_a < 0$

Too thin or too thick prisims are not preferred!

Same is also true for prisms with rectangular bases!

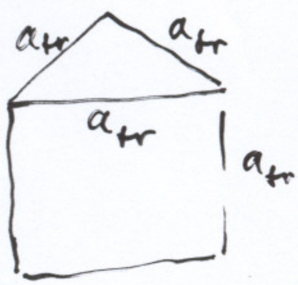
Discussions

- Three definitely positive energy results:
  spherical, conical and pyramidal cavities

  Compare the magnitudes of energies of these cavities for equal volumes:

$$\boxed{\begin{aligned} E_{pyr} &\simeq 0.51\, E_{sphere} \\ E_{con} &\simeq 0.54\, E_{sphere} \end{aligned}} \; ; \; E_{pry} \simeq E_{con}$$

Triangular prism with all edges are equal,



energy is positive $E_{tri} \simeq \dfrac{0.022}{a_{tr}} > 0$

To have same volume with sphere

$$a_{tr} \simeq 4.5\, a \Rightarrow E_{tri} \simeq 0.11\, E_{sphere}$$

Vertices absorbs energy!

Energy of cube with same volume $a_{cube} = \left(\frac{4\pi}{3}\right)^{1/3} a \simeq 3.5\, a$

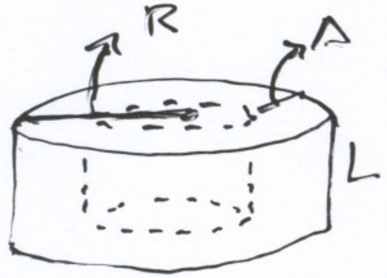
$$E_{cube} \simeq -\frac{0.004}{a} < 0 \quad \text{, close to zero!}$$

# of vertices		
	Pyra:	4
	Triang:	6
	Cube:	8

- Looks like ~~edges~~ too thin or thick forms are not preffered!

Results for some Multiply Connected Cavities

(i) Region between co-axial cylinders (close to each other



$$E_{cyl.} \simeq -\frac{\pi^3 R L}{720\Delta^3} + \frac{R\,\zeta(3)}{16\Delta^2}$$

$$\simeq -\frac{\pi^3 R L}{720\Delta^3} + \frac{0.08\, R}{\Delta^2} \quad (1)$$

For $\Delta = 0.1 R$, $E_{cyl} \simeq -\frac{42.9\, L}{R^2} + \frac{8}{R} < 0$

From (1): Energy is positive for $L \lesssim \frac{3}{2}\Delta$

For these heights radial force $F_{rad} = -\frac{\partial E}{\partial \Delta}$ is repulsive

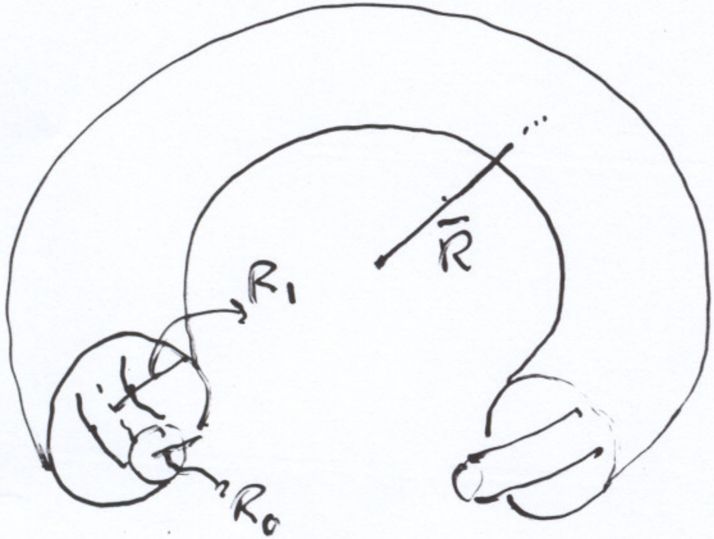
For $L > \frac{3}{2}\Delta$ radial force is attractive

Force in axial direction $F_{ax} = -\frac{\partial E}{\partial L}$ is always repulsive

$\Downarrow$

Long cylinders are preferred!

(ii) <u>Region between two (close) tori</u>



$$\Delta = R_1 - R_0 \ll R_0 \equiv R$$

$$E_{tor} = -\frac{\pi^3 R L}{720 \Delta^3}\left(1 + \frac{15}{2\pi^2}\frac{\Delta^2}{R^2}\right) < 0$$

always negative (L: circumferences of tori)

For $\Delta = 0.1 R$ $\qquad E_{tor} \simeq -\frac{5.4\, L}{R^2} < 0$.

Region between (close) cocentric spheres

$$\Delta = r_1 - r_0 \ll r_0$$

$$\mathcal{E}_{sphere} = -\frac{\pi^3 R^2}{360\,\Delta^3}\left(1 + \frac{5\Delta^2}{4\pi^2 R^2}\right) < 0$$

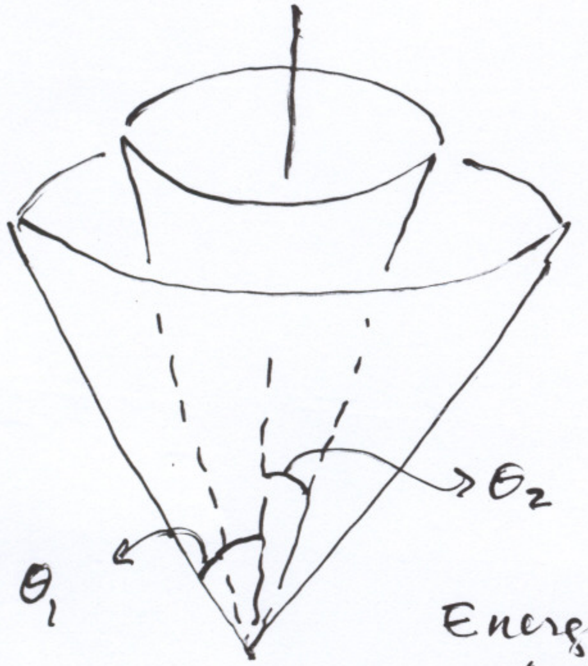
always negative

(approaches parallel plate formula in $\Delta/R \to 0$ limit)

For $\Delta = 0.1\,R$, $\mathcal{E}_{sphere} \simeq -88/R$ very big!

An open region result

Co-axial (close) cones

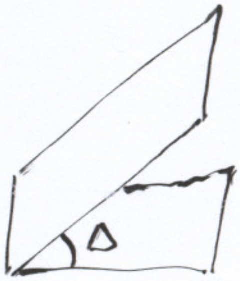


$$\Delta = \theta_1 - \theta_2 \ll \theta_1$$

Energy density at radial position r:

$$E_{con} \simeq -\frac{\pi^2}{1440 r^4 \Delta^4}$$

It is in agreement with energy density in region between planes with angle Δ (in $\Delta \to 0$ limit)



$$E_{wedge} = -\frac{1}{1440 r^4 \Delta^2}\left(\frac{\pi^2}{\Delta^2} - \frac{\Delta^2}{\pi^2}\right)$$