Matter Symmetries of Non-Static Plane Symmetric Spacetimes

M. Sharif *and Tariq Ismaeel
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus Lahore-54590, PAKISTAN.

Abstract

The matter collineations of plane symmetric spacetimes are studied according to the degenerate energy-momentum tensor. We have found many cases where the energy-momentum tensor is degenerate but the group of matter collineations is finite. Further we obtain different constraint equations on the energy-momentum tensor. Solving these constraints may provide some new exact solutions of Einstein field equations.

Keywords : Matter Collineations, Plane Symmetric Spacetimes

1 Introduction

General Theory of Relativity (GR), which is a field theory of gravitation and is described in terms of geometry, is highly non-linear. Because of this non-linearity, it becomes very difficult to solve the gravitational field equations unless certain symmetry restrictions are imposed on the spacetime metric. These symmetry restrictions are expressed in terms of isometries possessed by spacetimes. These isometries, which are also called Killing Vectors (KVs), give rise to conservation laws [1,2]. In GR, the Einstein tensor G_{ab} plays a significant role, since it relates the geometry of spacetime to its source. The GR theory, however, does not prescribe the various forms of matter, and takes over the energy-momentum tensor T_{ab} from other branches of physics.

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*msharif@math.pu.edu.pk

Einstein's field equations (EFEs) are given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}, \quad (a, b = 0, 1, 2, 3), \tag{1}$$

where G_{ab} are the components of the Einstein tensor, R_{ab} those of the Ricci and T_{ab} of the matter (energy-momentum) tensor. Also, $R = g^{ab}R_{ab}$ is the Ricci scalar, κ is the gravitational constant and for simplicity, we take $\Lambda = 0$.

Let (M, g) be a spacetime, i.e., M is a four-dimensional, Hausdorff, smooth manifold, and g is smooth Lorentz metric of signature (+ - - -) defined on M. The manifold M and the metric g are assumed smooth (C^∞). We shall use the usual component notation in local charts, and a covariant derivative with respect to the symmetric connection Γ associated with the metric g will be denoted by a semicolon and a partial derivative by a comma.

We define collineations as geometrical symmetries which are given by a relation of the form $\pounds_\xi A = B$, where $\pounds$ is the Lie derivative operator, ξ^a is the symmetry or collineation vector, A is any of the quantities $g_{ab}, \Gamma_{bc}^a, R_{ab}, R_{bcd}^a$ and geometric objects constructed by them and B is a tensor with the same index symmetries as A. One can find all the well-known collineations by requiring the particular forms of the quantities A and B. For example if we take $A_{ab} = g_{ab}$ and $B_{ab} = 2\psi g_{ab}$, this defines a Conformal Killing vector (CKV) and it specializes to a Special Conformal Killing vector (SCKAV) when $\psi_{;ab} = 0$, to a Homothetic vector (HV) field when $\psi = constant$ and to a Killing vector (KV) when $\psi = 0$. If we take $\Phi_{ab} = R_{ab}$ and $B_{ab} = 2\psi R_{ab}$ the symmetry vector ξ^a is called a Ricci Inheritance collineation (RIC) and reduces to a Ricci collineation (RC) for $B_{ab} = 0$. When $A_{ab} = T_{ab}$ and $B_{ab} = 2\psi T_{ab}$, where T_{ab} is the energy-momentum tensor, the vector ξ^a is called a Matter Inheritance collineation (MIC) and it reduces to a Matter collineation (MC) for $B_{ab} = 0$. In the case of CKVs, the function is called the conformal factor and in the case of inheriting collineations the inheriting factor.

We shall define MCs to be proper which is not a KV or a HV otherwise it is improper. The MC equation can be written as

$$\pounds_\xi T_{ab} = 0 \quad \Leftrightarrow \quad \pounds_\xi G_{ab} = 0, \tag{2}$$

or in component form

$$T_{ab,c}\xi^c + T_{ac}\xi_{,b}^c + T_{cb}\xi_{,a}^c = 0. \tag{3}$$

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A vector field ξ satisfying Eq.(2) or (3) on M is called a *matter collineatioin*.

There is a recent growing interest in the study of MCs [3-8]. Carot, et al. [4] have discussed MCs from the point of view of the Lie algebra of vector fields generating them and, in particular, they discussed spacetimes with a degenerate T_{ab}. Hall, et al. [5], in the discussion of RC and MC, have argued that the symmetries of the energy-momentum tensor may also provide some extra understanding of the the subject which has not been provided by Killing vectors, Ricci and Curvature collineations.

This paper has been meant to study the problem of calculating MCs for plane symmetric spacetimes when the energy-momentum tensor is degenerate only. A complete solution of the MC equations for the plane symmetric spacetimes will be reported elsewhere [9]. The paper has been organised as follows. In the next section we write down MC equations for plane symmetric spacetimes. In section three, we shall solve these MC equations when the energy-momentum tensor is degenerate only. We shall conclude the results at the end.

2 Matter Collineation Equations

This section contains the MC equations for plane symmetric spacetimes. The most general plane symmetric metric is given [10] as

$$ds^2 = e^{\nu(t,x)}dt^2 - e^{\lambda(t,x)}dx^2 - e^{\mu(t,x)}(dy^2 + dz^2). \tag{4}$$

The non-zero components of the energy-momentum tensor are T_{00}, T_{01}, T_{11}, T_{22}, T_3 given in Appendix A. MC Eqs.(3) for plane symmetric spacetime can be written as

$$T_{0,0}\xi^0 + T_{0,1}\xi^1 + 2T_0\xi^0_{,0} = 0, \tag{5}$$

$$T_0\xi^0_{,1} + T_1\xi^1_{,0} = 0, \tag{6}$$

$$T_0\xi^0_{,2} + T_2\xi^2_{,0} = 0, \tag{7}$$

$$T_0\xi^0_{,3} + T_2\xi^3_{,0} = 0, \tag{8}$$

$$T_{1,0}\xi^0 + T_{1,1}\xi^1 + 2T_1\xi^1_{,1} = 0, \tag{9}$$

$$T_1\xi^1_{,2} + T_2\xi^2_{,1} = 0, \tag{10}$$

$$T_1\xi^1_{,3} + T_2\xi^3_{,1} = 0, \tag{11}$$

$$T_{2,0}\xi^0 + T_{2,1}\xi^1 + 2T_2\xi^2_{,2} = 0, \tag{12}$$

$$T_2(\xi^2_{,3} + \xi^3_{,2}) = 0, \tag{13}$$

$$T_{2,0}\xi^0 + T_{2,1}\xi^1 + 2T_2\xi^3_{,3} = 0, \tag{14}$$

where $T_3 = T_2$. It is to be noticed that we are using the notation $T_{aa} = T_a$. Further, we have written these equations for higher symmetries under the assumption $T_{01} = 0$ which implies that either $\mu = constant$ or $\mu = \mu(x)$, $\lambda = \lambda(x)$. We shall solve these equations for the degenerate case only.

3 Solution of Matter Collineation Equations

In this section we solve MC equations (5)-(14) when the determinant of the energy-momentum is zero, i.e., $det(T_{ab}) = 0$. This means that we would require at least one of $T_a = 0$. First, we consider the trivial case, where $T_a = 0$. In this case, Eqs.(5)-(14) are identically satisfied and thus every direction is a MC.

The other possibilities can be classified in three main cases:

(1) when only one of $T_a \neq 0$,
(2) when two of $T_a \neq 0$,
(3) when three of $T_a \neq 0$.

However, we shall report only the case for which MCs are finite. This is the third case when three of $T_a \neq 0$. In this case, there could be only two possibilities, i.e., either

(3a) $T_0 = 0$, $T_i \neq 0$, $(i = 1, 2, 3)$

(3b) $T_1 = 0$, $T_j \neq 0$ $(j = 0, 2, 3)$.

We restrict ourselves to discuss the finite MCs of the first case only.
Case (3a): In this case, Eq.(5) is identically satisfied and Eqs.(6)-(8) respectively give $\xi^i = \xi^i(x, y, z)$. The remaining equations will become

$$T_{1,0}\xi^0 + T_{1,1}\xi^1 + 2T_1\xi^1_{,1} = 0, \tag{15}$$

$$T_1\xi^1_{,2} + T_2\xi^2_{,1} = 0, \tag{16}$$

$$T_1\xi^1_{,3} + T_2\xi^3_{,1} = 0, \tag{17}$$

$$T_{2,0}\xi^0 + T_{2,1}\xi^1 + 2T_2\xi^2_{,2} = 0, \tag{18}$$

$$\xi^2_{,3} + \xi^3_{,2} = 0, \tag{19}$$

$$T_{2,0}\xi^0 + T_{2,1}\xi^1 + 2T_2\xi^3_{,3} = 0. \tag{20}$$

From these equations, we have the following sixteen possibilities:

(i) $T_1 = constant \neq 0$, $T_2 = constant \neq 0$,

(ii) $T_{1,0} \neq 0$, $T_{1,1} = 0$, $T_2 = constant \neq 0$,

(iii) $T_{1,0} = 0$, $T_{1,1} \neq 0$, $T_2 = constant \neq 0$,

(iv) $T_1 = constant \neq 0$, $T_{2,0} \neq 0$, $T_{2,1} = 0$,

(v) $T_1 = constant \neq 0$, $T_{2,0} = 0$, $T_{2,1} \neq 0$,

(vi) $T_1 = constant \neq 0$, $T_{2,0} \neq 0$, $T_{2,1} \neq 0$,

(vii) $T_{1,0} \neq 0$, $T_{1,1} \neq 0$, $T_2 = constant \neq 0$,

(viii) $T_{1,0} \neq 0$, $T_{1,1} = 0$, $T_{2,0} \neq 0$, $T_{2,1} = 0$,

(ix) $T_{1,0} \neq 0$, $T_{1,1} = 0$, $T_{2,0} = 0$, $T_{2,1} \neq 0$,

(x) $T_{1,0} = 0$, $T_{1,1} \neq 0$, $T_{2,0} \neq 0$, $T_{2,1} = 0$,

(xi) $T_{1,0} = 0$, $T_{1,1} \neq 0$, $T_{2,0} = 0$, $T_{2,1} \neq 0$,

(xii) $T_{1,0} \neq 0$, $T_{1,1} \neq 0$, $T_{2,0} \neq 0$, $T_{2,1} = 0$,

(xiii) $T_{1,0} \neq 0$, $T_{1,1} = 0$, $T_{2,0} \neq 0$, $T_{2,1} \neq 0$,

(xiv) $T_{1,0} = 0$, $T_{1,1} \neq 0$, $T_{2,0} \neq 0$, $T_{2,1} \neq 0$,

(xv) $T_{1,0} \neq 0$, $T_{1,1} \neq 0$, $T_{2,0} = 0$, $T_{2,1} \neq 0$,

(xvi) $T_{1,0} \neq 0$, $T_{1,1} \neq 0$, $T_{2,0} \neq 0$, $T_{2,1} \neq 0$.

We list here only the finite cases.
Case(3aiv): Solving MC equations simultaneously, we obtain the following MCs

$$
\begin{aligned}
\xi^0 &= 0, \\
\xi^1 &= c_1 y + c_2 z + c_3, \\
\xi^2 &= -\frac{T_1}{T_2} c_1 x + c_4 z + c_5, \\
\xi^3 &= -\frac{T_1}{T_2} c_2 x - c_4 y + c_6.
\end{aligned} \tag{21}
$$

This gives six MCs out of which three are the usual KVs and three are the proper MCs.

Case(3avi): Solving MC equations under the constraints of this case, we obtain the following solution

$$
\begin{aligned}
\xi^0 &= -\frac{T_{2,1}}{T_{2,0}}[c_1 y + c_2 z + c_3], \\
\xi^1 &= c_1 y + c_2 z + c_3, \\
\xi^2 &= -c_1 \int \frac{T_1}{T_2} dx + c_4 z + c_5, \\
\xi^3 &= -c_2 \int \frac{T_1}{T_2} dx - c_4 y + c_6
\end{aligned}
\tag{22}
$$

giving rise to six MCs.

Case(3aviii): This gives the same results as the case (3aiv) and hence we obtain six MCs.

Case(3aix): Solving MC equations, after some algebraic manipulation, we obtain the following solution

$$
\begin{aligned}
\xi^\ell &= 0, \quad (\ell = 0, 1), \\
\xi^2 &= c_1 z + c_2, \\
\xi^3 &= -c_1 y + c_3.
\end{aligned}
\tag{23}
$$

In this case MCs turn out to be the usual three KVs.

Case(3ax): Proceeding in a similar way as above, it follows that MCs are

$$
\begin{aligned}
\xi^0 &= 0, \\
\xi^1 &= \frac{1}{\sqrt{T_1}}[c_1 y + c_2 z + c_3], \\
\xi^2 &= -\frac{c_1}{T_2} \int \sqrt{T_1} dx + c_4, \\
\xi^3 &= -\frac{c_2}{T_2} \int \sqrt{T_1} dx + c_5.
\end{aligned}
\tag{24}
$$

Here we get five MCs out of which two are proper.

Case(3axii): This case turns out to be similar to the case (3ax).

Case(3axiii): Here after some algebra, we have the following constraint

$$
\frac{A_i'(x)}{A_i(x)} = \frac{T_{1,0} T_{2,1}}{2 T_1 T_{2,0}} = \alpha,
$$

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where α is a separation constant. This gives rise to the following two possibilities:

$$(*) \quad \alpha = 0, \quad (**) \quad \alpha \neq 0.$$

Case(3axiii*): For $\alpha = 0$, MCs are

$$
\begin{aligned}
\xi^0 &= 0, \\
\xi^1 &= c_1 y + c_2 z + c_3, \\
\xi^2 &= -T_1 \int \frac{c_1}{T_2} dx + c_4 z + c_5, \\
\xi^3 &= -T_1 \int \frac{c_2}{T_2} dx - c_4 y + c_6.
\end{aligned}
\tag{25}
$$

It follows that we have six MCs.
Case(3axiii**): For $\alpha \neq 0$, MCs are

$$
\begin{aligned}
\xi^0 &= -\frac{T_{2,1}}{T_{2,0}} e^{\alpha x} [c_1 y + c_2 z + c_3], \\
\xi^1 &= e^{\alpha x} [c_1 y + c_2 z + c_3], \\
\xi^2 &= -c_1 \int \frac{T_1}{T_2} e^{\alpha x} dx + c_4 z + c_5, \\
\xi^3 &= -c_2 \int \frac{T_1}{T_2} e^{\alpha x} dx - c_4 y + c_6
\end{aligned}
\tag{26}
$$

giving six MCs.
Case(3axiv): Using the same procedure as above, after some algebra, we obtain the following MCs

$$
\begin{aligned}
\xi^0 &= -\frac{T_{2,1}}{T_{2,0}\sqrt{T_1}} [c_1 y + c_2 z + c_3], \\
\xi^1 &= \frac{1}{\sqrt{T_1}} [c_1 y + c_2 z + c_3], \\
\xi^2 &= -c_1 \int \frac{\sqrt{T_1}}{T_2} dx + c_4 z + c_5, \\
\xi^3 &= -c_2 \int \frac{\sqrt{T_1}}{T_2} dx - c_4 y + c_6.
\end{aligned}
\tag{27}
$$

Here again we obtain three proper MCs.
Case(3axvi): In this case, we further have the following constraint

$$\alpha = \frac{1}{2T_1} [T_{1,0} T_{2,1} - T_{1,1} T_{2,0}]$$

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giving the following two options

$$(*) \quad \alpha = 0, \quad (**) \quad \alpha \neq 0.$$

Case(3axvi*): This gives exactly the same results as the case (3avi).
Case(3axvi**): It gives the results similar to the case (3axiii**).

4 Discussion

In the classification of plane symmetric spacetimes according to the nature of the energy-momentum tensor, we find that when the energy-momentum tensor is degenerate, then there are many cases of MCs with infinite degrees of freedom. However, we have restricted to find the finite MCs only. It is very interesting to note that we have found many cases where the energy-momentum tensor is degenerate but the group of MCs are finite dimensional. We obtain three, five and six MCs out of which three are the usual KVs of the non-static plane symmetric spacetimes and rest are the proper MCs. In the cases (1)-(3), we summarize some results in the following:

1. In this case, the rank of T_{ab} being 1, it is found that all the possibilities yield infinite dimensional MCs.

2. In all subcases of this case, the rank of T_{ab} being 2.

3. In all subcases of this case, the rank of T_{ab} is 3. The worth mentioning point in this case are the subcases where we have finite dimensionality of the group of MCs even if the energy-momentum tensor is degenerate. We obtain *three, five and six* MCs.

We have obtained number of constraint equations. If these constraint equations could be solved, then one can expect to find new interesting exact solutions. A complete classification of the degenerate and non-degenerate energy-momentum tensor will be reported somewhere else [9].

Appendix A

The surviving components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= \frac{1}{4}(4\ddot{\nu} + 2\dot{\mu}^2 - \dot{\nu}\dot{\lambda} + 2\ddot{\lambda} + \dot{\lambda}^2 - 2\dot{\mu}\dot{\nu}) \\
&+ \frac{1}{4}e^{\nu-\lambda}(2\nu'' + \nu'^2 + 2\nu'\mu' - \nu'\lambda'), \\
R_{01} &= -\frac{1}{2}(2\dot{\mu}' + \dot{\mu}\mu' - \dot{\mu}\nu' - \dot{\lambda}\mu'), \\
R_{11} &= \frac{1}{4}e^{\lambda-\nu}(2\ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda} + 2\dot{\lambda}\dot{\mu}) \\
&- \frac{1}{4}(2\nu'' + \nu'^2 + 4\mu'' - \nu'\lambda' + 2\mu'^2 - 2\mu'\lambda'), \\
R_{22} &= \frac{1}{4}e^{\mu-\lambda}(2\ddot{\mu} + 2\dot{\mu}^2 - \dot{\nu}\dot{\mu} + \dot{\lambda}\dot{\mu}) \\
&- \frac{1}{4}e^{\mu-\lambda}(2\mu'' + 2\mu'^2 - \nu'\lambda' + \mu'\nu'), \\
R_{33} &= R_{22}. \tag{A1}
\end{aligned}
$$

The Ricci scalar is given by

$$
\begin{aligned}
R &= -\frac{1}{2}e^{-\nu}(2\ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda} - 2\dot{\nu}\dot{\mu} + 2\dot{\mu}\dot{\lambda} + 3\dot{\mu}^2 + 4\ddot{\mu}) \\
&+ \frac{1}{2}e^{-\lambda}(2\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'\mu' - 2\mu'\lambda' + 3\mu'^2 + 4\mu''). \tag{A2}
\end{aligned}
$$

Using Einstein field equations (1), the non-vanishing components of energy-momentum tensor T_{ab} are

$$
\begin{aligned}
T_{00} &= \frac{1}{4}(\dot{\mu}^2 + 2\dot{\mu}\dot{\lambda}) - \frac{1}{4}e^{\nu-\lambda}(4\mu'' + 3\mu'^2 - 2\mu'\lambda'), \\
T_{01} &= R_{01}, \\
T_{11} &= -\frac{1}{4}e^{\lambda-\nu}(4\ddot{\mu} + 3\dot{\mu}^2 - 2\dot{\mu}\nu) - \frac{1}{4}(\mu'^2 + 2\mu'\nu'), \\
T_{22} &= -\frac{1}{4}e^{\mu-\nu}(2\ddot{\mu} + 2\ddot{\lambda} + \dot{\mu}^2 - \dot{\mu}\dot{\nu} + \dot{\mu}\dot{\lambda} - \dot{\nu}\dot{\lambda} + \dot{\lambda}^2) \\
&+ \frac{1}{4}e^{\mu-\nu}(2\mu'' + 2\nu'' + \mu'^2 - \mu'\lambda' + \mu'\nu' - \lambda'\nu' + \nu'^2), \\
T_{33} &= T_{22}\sin^2\theta. \tag{A3}
\end{aligned}
$$

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