Electromagnetic and gravitational interactions of the spinning particle

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- Soon after Einstein proposed his gravitational theory, Weyl extended it to include electromagnetism and later revitalized his gauge idea, U(1) gauge invariance. In 1954, Yang and Mills generalized U(1) into SU(2). In 1956 Utiyama gauged the Lorentz group, SO(1,3), and later Kibble extended it into Poincare group, P .
- Utiyama R 1956, Phys. Rev. 101, 1597.
- Kibble T W B 1961, J. Math. Phys. 2 212.

In most of the gauge theories of gravitation, P is conceived as the spacetime symmetry group and the inner symmetry group which generate the translations and Lorentz rotations, and the SO(1, 3) frame rotations of matter fields, respectively.

- The global covariance of the matter field under P, yields the conservation of the energy-momentum and the total angular momentum.

In the local extension of the gauge group P, its spacetime part becomes the diffeomorphism group, the gauged theory is invariant under the general coordinate transformations and the local SO(1, 3) frame rotations.

In an other approach, P is considered as the internal symmetry group of matter fields in Minkowski spacetime to obtain a complementary gauge formulation of gravitation and to discuss the renormalization procedure.

- The Dirac equation is generalized into the curved spacetime by introducing the Fock
- - Ivananko 2-vector or the spin connection.

- Fock V A 1929 Zeits. Phys. 57 261 and Fock V A and Ivanenko D D 1929 Comptes Rendus des Seances de L'Academi des Sciences 188 1470.

- The Dirac algebra relates the metric tensor of the spacetime to the anti-commutator of the spacetime dependent Dirac matrices. In their pioneering investigations, Schrödinger and Bargmann discussed the generalization of the spin connection and showed that it gives the spacetime curvature, the spin-2 gravitational field, and an Abelian spin-1 curvature.

- Schrödinger E 1932 Sitz. Preuss. Ak. D. Wiss. 25 105. 13 : Bargmann V 1932 Sitz. Preuss. Ak. D. Wiss. 25 346.

- The aim of this study is to derive the minimal coupling of the electromagnetism and the gravitation with the spinning particle in a complemantary approach by performing the only spacetime transformations as the gauge transformations of the P and the internal coordinate transformations as the gauge transformations of the group U(2,2).

- First, we formulate the global Poincare transformations for the external spacetime coordinates, and the rotations and phase transformations between the holomorphic, internal coordinates of the spinning particle.
- The gauge groups of these transformations are P and U(2,2), respectively.

- The invariance of the free particle Lagrangian under the following subgroups of P and U(2,2) gives the conservation of the corresponding physical quantities:

The translation subgroup of P gives the conservation of the energy-momentum 1-vector.

The spacetime rotation subgroup of P with the internal rotation subgroup of U(2,2) together gives the conservation of the angular momentum 2-vector.

The internal U(1) phase transformation subgroup of U(2,2) gives the conservation of a scalar and real quantity, which will be identified as the electric charge of the particle.

- The free particle Lagrangian is also invariant under the global proper time translations and another real, non-negative scalar quantity is conserved, and this quantity will be identified as the mass of the particle.

- Second, we derive the gravitational and electromagnetic interactions of the spinnig particle as the local gauge transformations for the gauge groups P and U(2,2).

- We use the classical spinning particle model, developed to represent the classical analogue of the zitterbewegung motion of the electron. In this model, the particle has both usual spacetime degrees of freedom and four additional internal degrees of freedom.
- Proca A 1956 J. Phys. Radium 17 5 and Barut A O and Zanghi N 1984 Phys. Rev. Lett. 52 2009.

- To discuss the classical analogous of the zitterbewegung it is assumed that the internal dynamics of the particle consists of the four harmonic oscillators. The internal space is four dimensional Euclidian space and in dimensionless units the internal coordinates and momenta of the particle are q_i and π_i

$$z_i = \frac{1}{\sqrt{2}}\left(\frac{i\pi_i}{\sqrt{\epsilon p_0}} + \sqrt{\epsilon p_0}\, q_i\right)$$

$$z^{*i} = \frac{1}{\sqrt{2}}\left(\frac{-i\pi_i}{\sqrt{\epsilon p_0}} + \sqrt{\epsilon p_0}\, q_i\right)$$

Gauge transformations

- Global coordinate and phase transformations

$$x^a \rightarrow x'^a = x^a - i\xi^c P_c x^a - \frac{i}{2}\epsilon^{cd} L_{cd} x^a,$$

$$z_i \rightarrow z'_i = \exp\left(-i\phi 1 - \frac{i}{2}\epsilon^{cd}\Sigma_{cd}\right)_i{}^j z_j$$

$$\overline{z}^i \rightarrow \overline{z}'^i = \overline{z}^j \exp\left(+i\phi 1 + \frac{i}{2}\epsilon^{cd}\Sigma_{cd}\right)_j{}^i$$

$$\Sigma_{cd} = \frac{1}{2i}[\gamma_c, \gamma_d],$$

$$J_{ab} = L_{ab} + \frac{1}{2}\bar{z}\Sigma_{ab}z$$

$$s \to s' = s - i\tau H s,$$

$$H = p_a \bar{z} \gamma^a z.$$

Local coordinate and phase transformations

$$dx^a \rightarrow dx'^a = \left(\delta^a{}_b + \zeta^a{}_{,b} + \epsilon^a{}_{c,b}\, x^c \right) dx^b$$

$$z_i \to z'_i = U_i{}^j z_j$$

$$\bar{z}^{i} \rightarrow \bar{z}'^{i} = \bar{z}^{j}\left(U^{-1}\right)_{j}{}^{i}$$

$$U[x(s)] = \exp\left[-i\phi(x)1 - \frac{i}{4}\epsilon^{cd}(x)\Sigma_{cd}\right]$$

Under these transformations

$$\dot{x}^a \rightarrow \dot{x}'^a = \left(\delta^a{}_b + \zeta^a{}_{,b} + \epsilon^a{}_{c,b}\, x^c\right) \dot{x}^b \ ,$$

and

$$\bar{z}\gamma^a z \rightarrow \left(\bar{z}\gamma^a z\right)' = \bar{z}\gamma^a z.$$

$$p_a(\dot{x}^a - \bar{z}\gamma^a z) \rightarrow p_a e^a{}_\mu \left(\dot{x}^\mu - \bar{z} e^\mu{}_a \gamma^a z\right)$$

$$e^{a}{}_{\mu}(x) = \left(\delta^{a}{}_{\mu} + \zeta^{a}{}_{,\mu} + \epsilon^{a}{}_{c,\mu} x^{c}\right)$$

$$\gamma^{\mu}(x) = e^{\mu}{}_{a}(x)\gamma^{a}$$

$$p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z)$$

$$\frac{1}{2i}\left(\overline{z}\ z-\overline{z}\ \dot{z}\right)\rightarrow$$

$$\frac{1}{2i}\left(\dot{\overline{z}}\ z - \overline{z}\ \dot{z}\right) + \frac{1}{2i}\left(\overline{z}\,\frac{dU^{-1}}{ds}\,Uz - \overline{z}\,U^{-1}\,\frac{dU}{ds}z\right)$$

$$\frac{1}{2i}\left(\overline{z}\,\frac{dU^{-1}}{ds}\,Uz-\overline{z}\,U^{-1}\,\frac{dU}{ds}z\right)=\overline{z}\,B_{\mu}z\;\overline{z}\,\gamma^{\mu}z$$

$$B_{\mu} = \phi_{,\mu} + \frac{1}{4}\epsilon^{cd}{}_{,\mu}\Sigma_{cd}$$

$$L = \frac{1}{2i}\left(\overline{z}\; z - \overline{z}\; \dot{z}\right) + p_{\mu}\, \dot{x}^{\mu} - H$$

$$H = (p_\mu - \overline{z} B_\mu z) \overline{z} \gamma^\mu z = \Pi_\mu \overline{z} \gamma^\mu z$$

where

$$\overline{z} B_\mu z = \overline{z}(A_\mu + \Gamma_\mu) z$$

Equations of the motion

$$\frac{d}{ds}x^{\mu} = \dot{x}^{\mu} = \bar{z}\gamma^{\mu}z$$

$$\frac{D\bar{z}}{Ds} = \frac{d}{ds}\bar{z} + i\bar{z}B_{\mu}\dot{x}^{\mu} = i\bar{z}\gamma^{\alpha}\Pi_{\alpha},$$

$$\frac{Dz}{Ds} = \frac{d}{ds}z - iB_{\mu}z\dot{x}^{\mu} = -i\gamma^{\alpha}z\Pi_{\alpha},$$

$$\frac{D\,\dot{x}^{\mu}}{Ds} = \frac{d\,\dot{x}^{\mu}}{ds} + \dot{x}^{\alpha}\ g^{\mu\nu}\Gamma_{\alpha\nu\beta}\ \dot{x}^{\beta} = 4S^{\mu\nu}\Pi_{\nu} + \dot{x}^{\alpha}\ \bar{z}\gamma^{\mu}_{;\alpha}z$$

$$\gamma^{\mu}_{;\alpha} = \gamma^{\mu}{}_{,\alpha} + g^{\mu\nu}\Gamma_{\alpha\nu\beta}\gamma^{\beta}$$

$$\frac{DS^{\mu\nu}}{Ds} = \dot{x}^{\mu}\,\Pi^{\nu} - \dot{x}^{\nu}\,\Pi^{\mu} + \frac{1}{4i}\,\overline{z}\left[\gamma^{\mu}, \gamma^{\nu}\right]_{;\alpha} z\,\dot{x}^{\alpha}$$

$$\frac{d\Pi_\mu}{ds} = \dot{x}^\alpha \; c_{\alpha\;\mu}^{\;\;\beta} p_\beta - \bar{z}\left(\gamma^\alpha{}_{,\mu} + i[\gamma^\alpha, B_\mu]\right)z\Pi_\alpha + \dot{x}^\alpha \; \bar{z}\left(B_{\alpha,\mu} - B_{\mu,\alpha} + i[B_\alpha, B_\mu]\right)z,$$

where the commutator coefficients, $c_{\mu\nu}{}^\beta$ are

$$\frac{\partial p_\mu}{\partial x^\nu} - \frac{\partial p_\nu}{\partial x^\mu} = \left(\frac{\partial e^b_\mu}{\partial x^\nu} - \frac{\partial e^b_\nu}{\partial x^\mu}\right)p_b = c_{\mu\nu}{}^\beta p_\beta = \left(\Gamma_{\mu\nu}{}^\beta - \Gamma_{\nu\mu}{}^\beta - T_{\mu\nu}{}^\beta\right)p_\beta.$$

We define the covariant derivative of Π_μ by taking care of its vector character:

$$\frac{D\Pi_\mu}{Ds} = \frac{d\Pi_\mu}{ds} - \Gamma_{\mu\nu}^\beta \; \dot{x}^\nu \; \Pi_\beta$$

$$= -\dot{x}^\nu \; \left\{\left(\Gamma_{\mu\nu}{}^\beta - \Gamma_{\nu\mu}{}^\beta - c_{\mu\nu}{}^\beta\right)\Pi_\beta\right.$$

$$\left. - \bar{z}\left(B_{\nu,\mu} - B_{\mu,\nu} + i[B_\nu, B_\mu] - c_{\mu\nu}{}^\beta B_\beta\right)z\right\} - (\bar{z}\,\gamma^\nu_{;\mu} z)\Pi_\nu.$$

The torsion tensor, $T_{\mu\nu}{}^{\beta}$:

$$T_{\mu\nu}{}^{\beta} = \left(\Gamma_{\mu\nu}^{\beta} - \Gamma_{\nu\mu}^{\beta} - c_{\mu\nu}^{\beta}\right).$$

The curvature 2-form for the spin connection, $B_{\mu\nu}$ as

$$B_{\mu\nu} = \bar{z}\left(B_{\nu,\mu} - B_{\mu,\nu} + i[B_{\nu},\ B_{\mu}] - c_{\mu\nu}^{\beta} B_{\beta}\right)z.$$

$$B_{\mu\nu} = \bar{z}zF_{\mu\nu} + \frac{1}{2}\tilde{R}_{\alpha\beta\mu\nu}\frac{1}{2}\bar{z}\Sigma^{\alpha\beta}z,$$

The Lorentz force for the spinning particle in the electromagnetic and gravitational interactions is

$$\frac{D\Pi_{\mu}}{Ds} = -\dot{x}^{\alpha}\ T_{\mu\alpha}{}^{\beta}\Pi_{\beta} + \bar{z}zF_{\mu\nu}\ \dot{x}^{\nu} + \frac{1}{2}\tilde{R}_{\alpha\beta\mu\nu}\frac{1}{2}\bar{z}\Sigma^{\alpha\beta}z\ \dot{x}^{\nu} - (\bar{z}\gamma^{\nu}_{;\mu}z)\Pi_{\nu}.$$

Conservation of the currents

- The energy-momentum 1-vector.
- The angular momentum 2-vector.
- The electric current.

Conclusion

$\overline{z}z$ term is a constant of motion. Since it may be negative, zero or positive we identify it as the charge of the particle, e.

In the similar way, we also identify the other constant of the motion, the Hamiltonian, H as the mass energy of the particle, m.

We decompose Π_μ into to the parallel and normal components with respect to the velocity, $\dot{x}^\nu$:

$$\Pi_\mu = \frac{1}{(\dot{x})^2}\left(Hg_{\mu\nu}\ \dot{x}^\nu \ + \dot{x}^\nu\ \frac{DS_{\mu\nu}}{Ds}\right).$$

In this Eq. the first term of Π_μ is proportional $\dot{x}_\mu$ and the proportionality constant is the Hamiltonian, H and we identify it as the mass of the particle, m.

$$\frac{D}{Ds}\left[\left(mg_{\mu\nu}\ \dot{x}^\nu \ + \dot{x}^\nu\ \frac{DS_{\mu\nu}}{Ds}\right)\right] =$$

$$= -\ \dot{x}^\alpha\ T_{\mu\alpha}{}^{\beta}\left(mg_{\beta\nu}\ \dot{x}^\nu \ + \dot{x}^\nu\ \frac{DS_{\beta\nu}}{Ds}\right) + eF_{\mu\nu}\ \dot{x}^\nu \ + \frac{1}{2}\widetilde{R}_{\alpha\beta\mu\nu}\frac{1}{2}\bar{z}\Sigma^{\alpha\beta}z\ \dot{x}^\nu\ .$$

because of the zitterbewegung oscillations $(\dot{x})^2$ is not constant:

$$\dot{x}_\mu \frac{D\dot{x}^\mu}{Ds} = \frac{4\dot{x}_\mu S^{\mu\nu}}{\sqrt{(\dot{x})^2}} \frac{\dot{x}^\alpha}{\sqrt{(\dot{x})^2}} \frac{DS_{\nu\alpha}}{Ds} \neq 0.$$

In the phenemonological spin models $(\dot{x})^2$ is taken as a constant and then $p_\mu S^{\mu\nu}/\sqrt{(\dot{x})^2} = 0.$
Audretsch showed the vanishing of $D < \gamma^5\gamma^\mu > /Ds$ by using only the positive energy solutions of the Dirac equation. In our case the corresponding dynamical variables are

$$S^{5\mu} = \frac{1}{4i}\bar{z}[i\gamma^5, \gamma^\mu]z,$$

and their proper time dependence is given as

$$\frac{DS^{5\mu}}{Ds} = \dot{x}^5\ \Pi^\mu,$$

where $\dot{x}^5$ is $\bar{z}i\gamma^5 z.$ We think that the difference between these two results comes from the contribution of the zitterbewegung oscillations.

For the spinless particle $S^{\alpha\beta}$ vanishes, $(\dot{x})^2 = 1$, and $\Pi_\mu = mg_{\mu\nu}\,\dot{x}^\nu$. Then the Lorentz force becomes

$$\frac{D}{Ds}(g_{\mu\nu}\,\dot{x}^\nu) = -T_{\mu\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta + \frac{e}{m}F_{\mu\nu}\,\dot{x}^\nu$$

for the Riemann-Cartan space, U_4. For the Riemann space, V_4 the torsion tensor is zero and we obtain the geodesic equation with the electromagnetic force. For the Weitzenbock space, W_4, the curvature is zero and we obtain the nongeodesic equation with Cartan connection.