The derivative of the topological susceptibility at zero momentum and an estimate of η' mass in the chiral limit

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Abstract

The anomaly-anomaly correlator is studied using QCD sum rules. Using the matrix elements of anomaly between vacuum and pseudoscalars π, η and η', the derivative of correlator $\chi'(0)$ is evaluated and found to be $\approx 1.82 \times 10^{-3}$ GeV^2. Assuming that $\chi'(0)$ has no significant dependence on quark masses, the mass of η' in the chiral limit is found to be $\approx$723 MeV. The same calculation also yields for the singlet pseudoscalar decay constant in the chiral limit a value of $\approx$ 178 MeV.

The axial vector current in QCD has an anomaly

$$\partial^\mu \bar{q}\gamma_\mu\gamma_5 q = 2\, i\; m_q\; \bar{q}\gamma_5 q - \frac{\alpha_s}{4\pi}\, G^a_{\mu\nu}\tilde{G}^{a\mu\nu}, \quad \text{where, } \tilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}. \tag{1}$$

The topological susceptibility $\chi(q^2)$ defined by

$$\chi(q^2) = i \int d^4x\, e^{iq.x} \langle 0| T\,\{Q(x),\; Q(0)\}\,|0\rangle, \quad \text{with, } Q(x) = \frac{\alpha_s}{8\pi}\, G^a_{\mu\nu}\tilde{G}^{a\mu\nu} \tag{2}$$

$$\chi'(0) = \frac{d\chi(q^2)}{dq^2}\Big|_{q^2=0}$$

Using desperssion relation one can write

$$\frac{\chi'(q^2)}{q^2} - \frac{\chi'(0)}{q^2} = \frac{1}{\pi}\int ds\, \Im(\chi(s)) \left[\frac{1}{s(s-q^2)^2} + \frac{1}{s^2(s-q^2)}\right] + \text{subtractions.}$$

Defining the Borel transform of a fuction $f(q^2)$ by

$$\hat{B} f(q^2) = \lim_{-q^2, n \to \infty} \left[\frac{(-q^2)^{n+1}}{n!} \left(\frac{d}{dq^2}\right)^n f(q^2)\right]_{-q^2/n = M^2}$$

one gets from Eq.(4)

$$\chi'(0) = \frac{1}{\pi}\int ds\, \frac{\Im(\chi(s))}{s^2} \left(1 + \frac{s}{M^2}\right) e^{-s/M^2} - \hat{B}\left[\frac{\chi'(q^2)}{q^2}\right]. \tag{6}$$

$\Im(\chi(s))$ receives contribution from all states $|n\rangle$ such that $\langle 0|Q|n\rangle \neq 0$.

$$\langle 0|Q|\pi^0\rangle = i\, f_\pi\, m_\pi^2 \left(\frac{m_d - m_u}{m_d + m_u}\right) \frac{1}{2\sqrt{2}}. \tag{7}$$

The matrix elements, when $|n\rangle$ is $|\eta\rangle$ or $|\eta'\rangle$, can be determined as follows. It is known theoretical considerations based on chiral perturbation theory as well as phenomenological an one needs two mixing angles θ_8 and θ_0 to describe the coupling of the octet and siglet ax currents to η and η' [7, 8, 9]. Introduce the definition

$$\langle 0|J^a_{\mu 5}|P(p)\rangle = i\, f^a_P\, p_\mu;\ a = 0, 8;\ P = \eta, \eta',$$

$$J^8_{\mu 5} = \frac{1}{\sqrt{6}} (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s) \tag{9}$$

$$J^0_{\mu 5} = \frac{1}{\sqrt{3}} (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s)\,. \tag{10}$$

The $|P(p)\rangle$ represents either η or η' with momentum p_μ.

$$\begin{pmatrix} f^8_\eta & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{pmatrix} = \begin{pmatrix} f_8\ \cos\theta_8 & -f_0\ \sin\theta_0 \\ f_8\ \sin\theta_8 & f_0\ \cos\theta_0 \end{pmatrix} \tag{11}$$

Escribano and Frere find,

with $\qquad f_8 = 1.28 f_\pi \; (f_\pi = 130.7\text{MeV}),$

the other three parameters to be

$$\theta_8 = (-22.2 \pm 1.8)^\circ, \quad \theta_0 = (-8.7 \pm 2.1)^\circ, \quad f_0 = (1.18 \pm 0.04)\; f_\pi .$$

The divergence of the axial currents are given by

$$\partial^\mu J^8_{\mu 5} = \frac{i\,2}{\sqrt{6}} \left(m_u\, \bar{u}\gamma_5 u + m_d\, \bar{d}\gamma_5 d - 2m_s\, \bar{s}\gamma_5 s\right) \tag{14}$$

$$\partial^\mu J^0_{\mu 5} = \frac{i\,2}{\sqrt{3}} \left(m_u\, \bar{u}\gamma_5 u + m_d\, \bar{d}\gamma_5 d + m_s\, \bar{s}\gamma_5 s\right) + \frac{1}{\sqrt{3}}\,\frac{3\alpha_s}{4\pi}\, G^a_{\mu\nu}\tilde{G}^{a\mu\nu} \tag{15}$$

Since $m_u, m_d << m_s$, one can neglect them [10] to obtain

$$\langle 0|\frac{3\alpha_s}{4\pi} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}|\eta\rangle = \sqrt{\frac{3}{2}}\, m^2_\eta \left(f_8\, \cos\theta_8 - \sqrt{2} f_0\, \sin\theta_0\right) \tag{16}$$

$$\langle 0|\frac{3\alpha_s}{4\pi} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}|\eta'\rangle = \sqrt{\frac{3}{2}}\, m^2_{\eta'} \left(f_8\, \sin\theta_8 + \sqrt{2} f_0\, \cos\theta_0\right). \tag{17}$$

Using Eqs.(7), (16) and (17) we get the representation of $\chi(q^2)$ in terms of physical states as

$$\chi(q^2) = -\frac{m^4_\pi}{8(q^2 - m^2_\pi)}\, f^2_\pi \left(\frac{m_d - m_u}{m_d + m_u}\right)^2 - \frac{m^4_\eta}{24(q^2 - m^2_\eta)} \left(f_8\, \cos\theta_8 - \sqrt{2} f_0\, \sin\theta_0\right)^2$$
$$-\frac{m^4_{\eta'}}{24(q^2 - m^2_{\eta'})} \left(f_8\, \sin\theta_8 + \sqrt{2} f_0\, \cos\theta_0\right)^2 + \text{higher mass states.} \tag{18}$$

On the other hand, $\chi(q^2)$ has an operator product expansion [11, 12, 1, 5]

$$
\begin{aligned}
\chi(q^2)_{OPE} \quad = \quad & -\left(\frac{\alpha_s}{8\pi}\right)^2 \frac{2}{\pi^2}\, q^4 \ln\left(\frac{-q^2}{\mu^2}\right) \left[1 + \frac{\alpha_s}{\pi} \left(\frac{83}{4} - \frac{9}{4} \ln\left(\frac{-q^2}{\mu^2}\right)\right)\right] \\
& - \frac{1}{16} \frac{\alpha_s}{\pi} \langle 0| \frac{\alpha_s}{\pi} G^2 |0\rangle \left(1 - \frac{9}{4} \frac{\alpha_s}{\pi} \ln\left(\frac{-q^2}{\mu^2}\right)\right) + \frac{1}{8q^2} \frac{\alpha_s}{\pi} \langle 0| \frac{\alpha_s}{\pi} g_s G^3 |0\rangle \\
& - \frac{15}{128} \frac{\pi \alpha_s}{q^4} \langle 0| \frac{\alpha_s}{\pi} G^2 |0\rangle^2 + 16 \left(\frac{\alpha_s}{4\pi}\right)^3 \sum_{i=u,d,s} m_i \langle \bar{q}_i q_i \rangle \left[\ln\left(\frac{-q^2}{\mu^2}\right) + \frac{1}{2}\right] \\
& - \left[\frac{q^4}{2} \int d\rho\, n(\rho)\, \rho^4\, K_2^2(Q\rho) + \text{screening correction to the direct instantons}\right] . \quad (19)
\end{aligned}
$$

From Eq.(6), we now obtain

$$
\begin{aligned}
\chi'(0) \;=\; & \frac{f_\pi^2}{8}\left(\frac{m_d-m_u}{m_d+m_u}\right)^2\left(1+\frac{m_\pi^2}{M^2}\right)\, e^{\frac{-m_\pi^2}{M^2}} + \frac{1}{24}\left(f_8\cos\theta_8-\sqrt{2}f_0\sin\theta_0\right)^2\,\left(1+\frac{m_\eta^2}{M^2}\right)\, e^{\frac{-m_\eta^2}{M^2}} \\
& + \frac{1}{24}\left(f_8\sin\theta_8+\sqrt{2}f_0\cos\theta_0\right)^2\,\left(1+\frac{m_{\eta'}^2}{M^2}\right)\, e^{\frac{-m_{\eta'}^2}{M^2}} \\
& - \left(\frac{\alpha_s}{4\pi}\right)^2\frac{1}{\pi^2}\, M^2\, E_0(W^2/M^2)\left[1+\frac{\alpha_s}{\pi}\frac{74}{4}+\frac{\alpha_s}{\pi}\frac{9}{2}\left(\gamma-\ln\frac{M^2}{\mu^2}\right)\right] \\
& - 16\left(\frac{\alpha_s}{4\pi}\right)^3\,\frac{1}{M^2}\sum_{i=u,d,s} m_i\langle\bar{q}_i q_i\rangle - \frac{9}{64}\frac{1}{M^2}\left(\frac{\alpha_s}{\pi}\right)^2\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle \\
& + \frac{1}{16}\frac{1}{M^4}\frac{\alpha_s}{\pi}\left\langle g_s\frac{\alpha_s}{\pi}G^3\right\rangle - \frac{5}{128}\frac{\pi^2}{M^6}\frac{\alpha_s}{\pi}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle^2 .
\end{aligned} \tag{20}
$$

Here $E_0(x) = 1 - e^{-x}$ and takes into account the contribution of higher mass states, which has been summed using duality to the perturbative term in χ_{OPE}, and W is the effective continuum threshold. We take $W^2 = 2.3$ GeV^2, and Fig.1 plot the r.h.s. of Eq.(20) as a function of M^2. We take $\alpha_s = 0.5$ for $\mu = 1$ GeV and

$$\langle 0|g_s^2 G^2|0\rangle = 0.5 \text{ GeV}^2,\ \langle 0|\bar{s}s|0\rangle = 0.8\langle 0|\bar{u}u|0\rangle \text{ with } \langle 0|\bar{u}u|0\rangle = -(240 \text{ MeV})^3,$$
$$m_s = 150 \text{ MeV and } m_u/m_d \approx 0.5. \tag{21}$$

Writing

$$\langle 0|g_s^3\, G^3|0\rangle = \frac{\epsilon}{2}\langle 0|g_s^2\, G^2|0\rangle, \tag{22}$$

as in Ref. [5], we take $\epsilon = 1$ GeV^2. We also have the PCAC relation,

$$-2(m_u + m_d)\ \langle 0|\bar{u}u|0\rangle = f_\pi^2\, m_\pi^2. \tag{23}$$

For f_0, f_8, θ_8 and θ_0 we use the central values given in Eqs.(12) and (13).

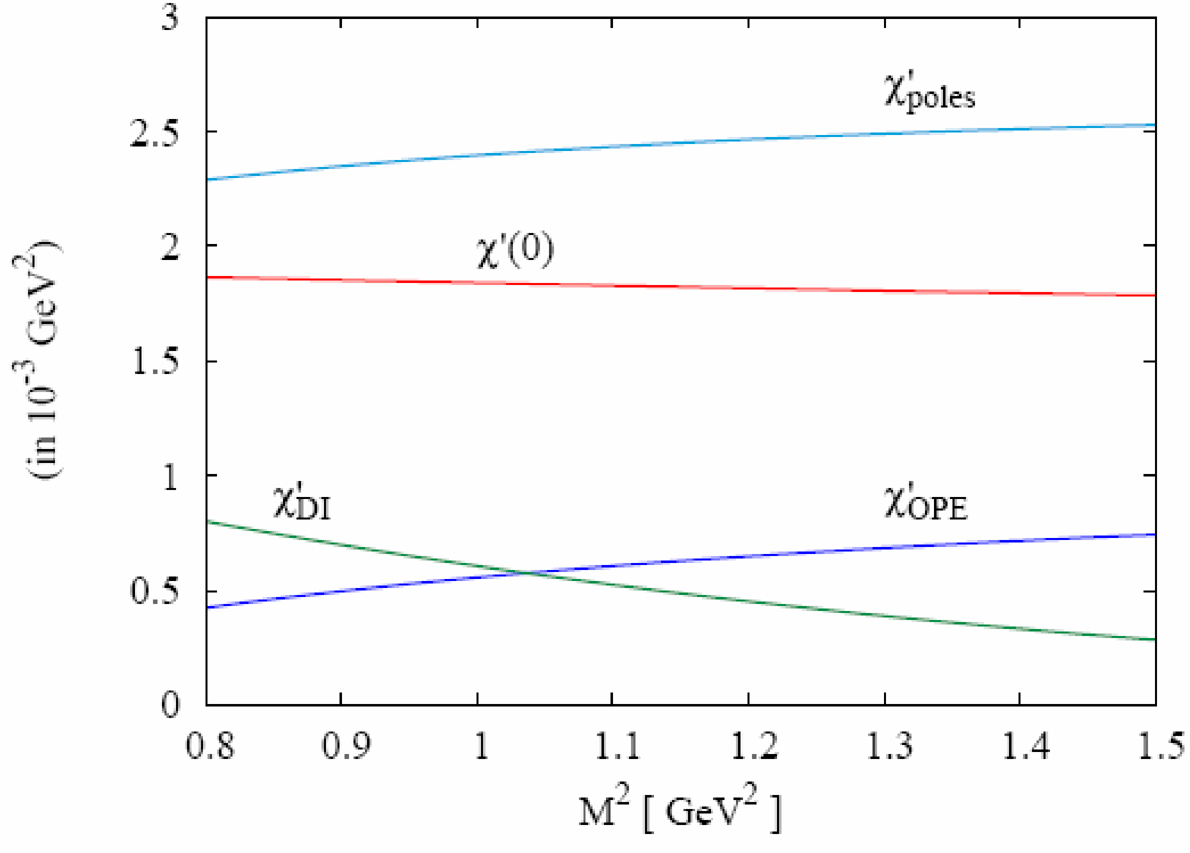

Fig. 1: Various ters contributing to $\chi'(0)$, Eq.(20). The value of $\chi'(0)$ is the one obtained without the direct instantons. The latter, see Eq.(29), is given by χ'_{DI}, which is larger than χ'_{OPE} and also has the wrong M^2 behaviour suggesting that screening corrections are important.

$$\chi'(0) \approx 1.82 \times 10^{-3}\ \mathrm{GeV}^2.$$

We note that the above determination, Eq.(24), is in agreement with an entirely different calculation by two of us [14] from the study of the correlator of isoscalar axial vector currents

$$\begin{aligned}\Pi^{I=0}_{\mu\nu} &= \frac{i}{2}\int d^4x\, e^{iq.x}\langle 0|\left\{\bar{u}\gamma_\mu\gamma_5 u(x) + \bar{d}\gamma_\mu\gamma_5 d(x), \bar{u}\gamma_\mu\gamma_5 u(0) + \bar{d}\gamma_\mu\gamma_5 d(0)\right\}|0\rangle \\ \Pi^{I=0}_{\mu\nu} &= -\Pi^{I=0}_1(q^2)\, g_{\mu\nu} + \Pi^{I=0}_2(q^2) q_\mu q_\nu . \end{aligned} \tag{25}$$

$\Pi^{I=0}_1(q^2 = 0)$ can be computed from the spectrum of axial vector mesons. In Ref. [14] a value

$$\Pi^{I=0}_1(q^2 = 0) = -0.0152\ \mathrm{GeV}^2$$

It is not difficult to see that when $m_u = m_d = 0$

$$\chi'(0) = -\frac{1}{8}\,\Pi^{I=0}_1(q^2 = 0)$$

On the other hand, $\chi(q^2)$ has an operator product expansion [11, 12, 1, 5]

$$\begin{aligned}
\chi(q^2)_{OPE} \quad = \quad & -\left(\frac{\alpha_s}{8\pi}\right)^2 \frac{2}{\pi^2}\, q^4 \, \ln\left(\frac{-q^2}{\mu^2}\right) \left[1+\frac{\alpha_s}{\pi}\left(\frac{83}{4}-\frac{9}{4}\ln\left(\frac{-q^2}{\mu^2}\right)\right)\right] \\
& -\frac{1}{16}\,\frac{\alpha_s}{\pi}\,\langle 0|\frac{\alpha_s}{\pi}G^2|0\rangle \left(1-\frac{9}{4}\frac{\alpha_s}{\pi}\,\ln\left(\frac{-q^2}{\mu^2}\right)\right)+\frac{1}{8q^2}\frac{\alpha_s}{\pi}\,\langle 0|\frac{\alpha_s}{\pi}g_s G^3|0\rangle \\
& -\frac{15}{128}\frac{\pi\alpha_s}{q^4}\,\langle 0|\frac{\alpha_s}{\pi}G^2|0\rangle^2+16\left(\frac{\alpha_s}{4\pi}\right)^3 \sum_{i=u,d,s} m_i \,\langle \bar{q}_i q_i\rangle \left[\ln\left(\frac{-q^2}{\mu^2}\right)+\frac{1}{2}\right] \\
& -\left[\frac{q^4}{2}\int d\rho\, n(\rho)\,\rho^4\, K_2^2(Q\rho)+\text{screening correction to the direct instantons}\right].
\end{aligned} \quad (19)$$

$$n(\rho) = n_0\, \delta(\rho - \rho_c) \tag{28}$$

with $n_0 = 0.75 \times 10^{-3}$ GeV^4 and $\rho_c = 1.5$ GeV^{-1}. The contribution of the direct instanton to $\hat{B}[\chi'(q^2)/q^2]$ can be found using the asymptotic expansion for $K_2(z)$ and $K_2'(z)$ and we find it to be

$$\chi'_{DI} = \frac{n_0}{4}\, \sqrt{\pi}\, \rho_c^4\, M^2 \left[M\rho_c + \frac{9}{4}\frac{1}{M\rho_c} + \frac{45}{32}\frac{1}{M^3\rho_c^3} \right] e^{-M^2\rho_c^2}. \tag{29}$$

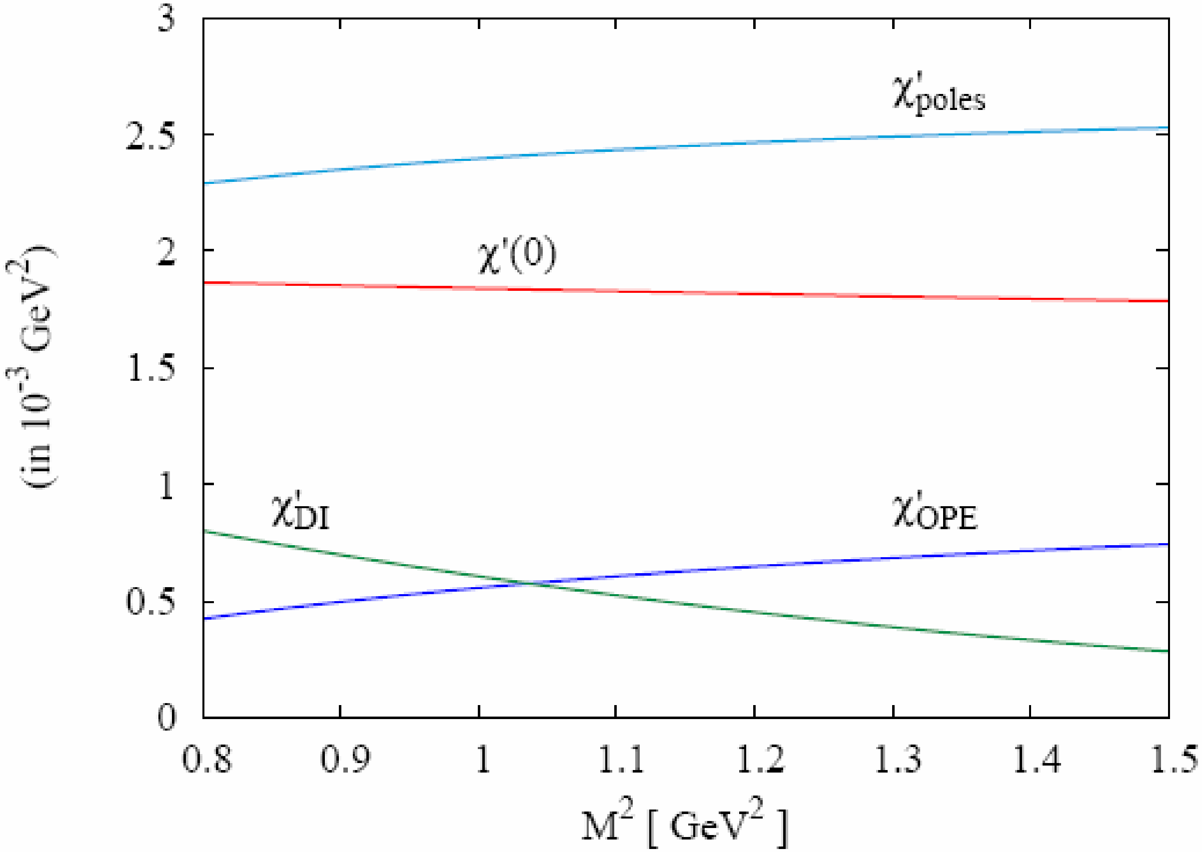

Fig. 1: Various ters contributing to $\chi'(0)$, Eq.(20). The value of $\chi'(0)$ is the one obtained without the direct instantons. The latter, see Eq.(29), is given by χ'_{DI}, which is larger than χ'_{OPE} and also has the wrong M^2 behaviour suggesting that screening corrections are important.

We now turn to an estimate of η' mass in the chiral limit: $m_u = m_d = m_s = 0$.

$SU(3)$ flavor symmetry is exact and, we have $m_\pi = m_\eta = 0$ while η' is a singlet.

Let us denote by

$$\eta_\chi = \eta'(m_s = 0) \text{ and } m_\chi = m_{\eta'}(m_s = 0),$$

we first note that the explicitly quark mass dependent term in χ_{OPE}

$$-16\left(\frac{\alpha_s}{4\pi}\right)^3 \sum_{i=u,d,s} m_i \langle \bar{q}_i q_i \rangle \approx 1.9 \times 10^{-6}\ \mathrm{GeV}^4$$

is numerically much smaller than, for expample

$$\frac{9}{64}\left(\frac{\alpha_s}{\pi}\right)^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \approx 4.5 \times 10^{-5}\ \mathrm{GeV}^4$$

which itself is much smaller than the perturbative term.

In the chiral limit $\langle 0|Q|\pi\rangle = \langle 0|Q|\eta\rangle = 0$. If we

assume that the quark mass dependence of $\chi'(0)$ is negligible then $\chi'(0)$ in Eq.(20) can also be expressed in term of f_{η_χ} and m_χ as:

$$\chi'(0) = \frac{1}{12} f_{\eta_\chi}^2 \left(1 + \frac{m_\chi^2}{M^2}\right) e^{\frac{-m_\chi^2}{M^2}} - \hat{B}\left[\frac{\chi'_{OPE}(q^2)}{q^2}\right].$$

$$\begin{aligned}
\frac{1}{12} f_{\eta_\chi}^2 \left(1 + \frac{m_\chi^2}{M^2}\right) e^{\frac{-m_\chi^2}{M^2}} &\approx \frac{1}{24} f_\pi^2 \left(1 + \frac{m_\pi^2}{M^2}\right) e^{\frac{-m_\pi^2}{M^2}} \\
&+ \frac{1}{24}\left(f_8 \cos\theta_8 - \sqrt{2} f_0 \sin\theta_0\right)^2 \left(1 + \frac{m_\eta^2}{M^2}\right) e^{\frac{-m_\eta^2}{M^2}} \\
&+ \frac{1}{24}\left(f_8 \sin\theta_8 + \sqrt{2} f_0 \cos\theta_0\right)^2 \left(1 + \frac{m_{\eta'}^2}{M^2}\right) e^{\frac{-m_{\eta'}^2}{M^2}}.
\end{aligned}$$

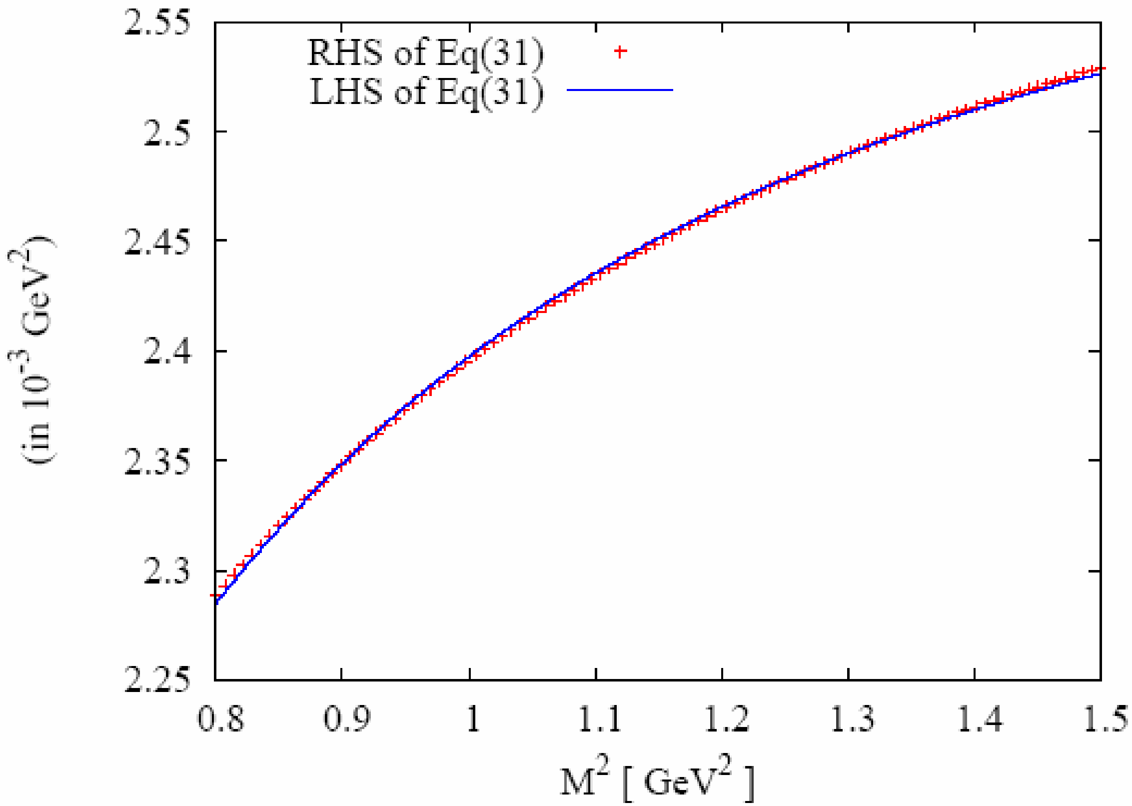

Fig. 2: Estimate of η' mass and coupling in the chiral limit, see Eq.(31). The continuous curve corresponds to $m_\chi = 723$ MeV.

We find $m_\chi \approx 723$ MeV and corresponding $f_{\eta\chi} = 178$ MeV.