Ricci Collineations in Bianchi II Spacetime

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A one-parameter group of conformal motions generated by a *conformal Killing vector* (CKV) ξ is defined by [1]

$$\pounds_\xi g_{ab} = 2\psi g_{ab} \Longleftrightarrow g_{ab,c}\xi^c + g_{ac}\xi^c_{,b} + g_{cb}\xi^c_{,a} = 2\psi g_{ab}, \qquad (1)$$

where $\pounds_\xi$ is the Lie derivative operator along the vector field ξ, and $a, b, c, ... = 0, 1, 2, 3$; $\psi = \psi(x^a)$ is a conformal factor. If $\psi_{;ab} \neq 0$, the CKV is said to be proper.

$$\begin{aligned} \psi_{;ab} = 0 \quad &\Longleftrightarrow \quad \xi \text{ special conformal Killing vector(SCKV)} \\ \psi_{,a} = 0 \quad &\Longleftrightarrow \quad \xi \text{ homothetic vector(HV)} \\ \psi = 0 \quad &\Longleftrightarrow \quad \xi \text{ Killing vector(KV)} \end{aligned}$$

In most situations of physical interest, we have spacetime symmetries which further reduce the number of unknown functions [2].

The Einstein field equations (EFEs), which are a set of coupled non-linear partial differential equations,

$$(G_{ab} \equiv) R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}, \qquad (2)$$

ten unknown functions g_{ab} when $R_{ab} = 0$ $+$

{ the mass-energy density ρ, pressure p , ... } when $T_{ab} \neq 0$

The well-known symmetry of the Ricci tensor is called as the *Ricci collineation* (RC) defined by [1],

$$\pounds_{\mathbf{X}} R_{ab} = 0 \Longleftrightarrow R_{ab,c} X^c + R_{ac} X^c_{,b} + R_{cb} X^c_{,a} = 0, \qquad (3)$$

where $\mathbf{X} = X^a \frac{\partial}{\partial x^a}$ is the vector field generating the RC symmetry.

Recently, much interest has been shown in the study of *Matter collineations* (MCs) defined by

$$\pounds_{\mathbf{Y}} T_{ab} = 0 \Longleftrightarrow T_{ab,c} Y^c + T_{ac} Y^c_{,b} + T_{cb} Y^c_{,a} = 0. \qquad (4)$$

When we assume the EFEs, the vector field $\mathbf{Y}$ generates an *Einstein collineation* if

$$\pounds_{\mathbf{Y}} G_{ab} = 0 \Longleftrightarrow \pounds_{\mathbf{Y}} T_{ab} = 0$$

The MCs and the RCs of the FRW metric have been studied by Camci and Barnes [3]. Tsamparlis and Apostolopoulos [4] have determined the RCs of Bianchi I space-time in the case of non-degenerate Ricci tensor.

Camci and his collaborators[5], [6] have classified the RCs of Kantowski-Sachs, Bianchi I and III spacetimes. A family of RCs of Bianchi II, VIII, and IX spacetimes have been discussed by Yavuz and Camci [7]. The RCs and MCs of locally rotationally symmetric spacetimes are presented in [8]. Recently, we have classified the RCs in perfect fluid Bianchi V spacetime [9]. Here we provide a complete RC classification of the Bianchi II spacetime according to the nature of the Ricci tensor R_{ab}.

Some important results about the Lie algebra of RCs[11]:

a. The set of all RCs on manifold M is a vector space, but it may be infinite dimensional and may not be a Lie algebra. If R_{ab} is non-degenerate, i.e. $det(R_{ab}) \neq 0$, the Lie algebra of RCs is finite dimensional. If R_{ab} is degenerate, i.e. $det(R_{ab}) = 0$, we cannot guarantee the finite dimensionality of the RCs.

b. If R_{ab} is everywhere of rank 4 then it may be regarded as a metric on manifold. Then, it comes out as a standard result that the family of RCs is, in fact, a Lie algebra of smooth vector fields on manifold M of finite dimension ≤ 10 (and $\neq 9$).

The line element for the spatially homogeneous Bianchi II spacetime is of the form [2],[12]

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 \left[dy - x dz\right]^2 + C^2 dz^2, \tag{5}$$

where $A = A(t), B = B(t)$, and $C = C(t)$.

This spacetime admits a group of isometries G_3, acting on spacelike hypersurfaces, generated by the spacelike KVs

$$\xi_{(1)} = \partial_y, \quad \xi_{(2)} = \partial_z, \quad \xi_{(3)} = \partial_x + z\partial_y. \tag{6}$$

The Lie algebra has the following non-diagonal commutators:

$$\left[\xi_{(1)}, \xi_{(2)}\right] = 0, \quad \left[\xi_{(1)}, \xi_{(3)}\right] = 0, \quad \left[\xi_{(2)}, \xi_{(3)}\right] = \xi_{(1)}. \tag{7}$$

The non-vanishing components of the R_{ab} are given by

$$R_{00} \equiv R_0 = -\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right), \tag{8}$$

$$R_{11} \equiv R_1, \quad R_{22} \equiv R_2, \tag{9}$$

$$R_{23} = -x R_2, \quad R_{33} \equiv x^2 R_2 + f, \tag{10}$$

where $R_1(t), R_2(t)$ and $f(t)$ are defined as

$$R_1 = A^2\left(\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - \frac{B^2}{2A^2C^2}\right), \tag{11}$$

$$R_2 = B^2\left(\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{B^2}{2A^2C^2}\right), \tag{12}$$

$$f = C^2\left(\frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{B^2}{2A^2C^2}\right), \tag{13}$$

where the *dot* denotes derivative with respect to t. Then, we find the scalar curvature R as

$$R = 2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) + 2\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) - \frac{B^2}{2A^2C^2}. \tag{14}$$

The *Ricci tensor metric* for Bianchi II spacetime is given by

$$ds^2_{Ric} = R_0\, dt^2 + R_1\, dx^2 + R_2\left[dy - x dz\right]^2 + f\, dz^2 \tag{15}$$

where R_0, R_1, R_2, and f are given the above. Here it is obviously seen that this metric has the original Bianchi II form given by (5). Thus, the signature of the Ricci tensor metric depends on the signs of R_0, R_1, R_2, f, and is Lorentzian if R_0 and the others R_1, R_2, f have opposite signs and is positive or negative definite if they have same sign.

For the Bianchi II spacetime (5), using the non-zero Ricci tensor components (8)-(10), we can write the RC equations (3), generated by an arbitrary vector field $X^a(t,x,y,z)$, in terms of $R_a(t)$ as follows:

$$\dot{R}_0 X^0 + 2R_0 X^0_{,t} = 0, \tag{16}$$

$$\dot{R}_1 X^0 + 2R_1 X^1_{,x} = 0, \tag{17}$$

$$\dot{R}_2 X^0 + 2R_2 F_{,y} = 0, \tag{18}$$

$$\left(x^2 \dot{R}_2 + \dot{f}\right) X^0 + 2xR_2 \left(X^1 - F_{,z}\right) + 2f\, X^3_{,z} = 0, \tag{19}$$

$$x\dot{R}_2 X^0 + R_2 \left(X^1 - F_{,z} + xF_{,y}\right) - f\, X^3_{,y} = 0, \tag{20}$$

$$R_0 X^0_{,x} + R_1 X^1_{,t} = 0, \tag{21}$$

$$R_0 X^0_{,y} + R_2 F_{,t} = 0, \tag{22}$$

$$R_0 X^0_{,z} - xR_2 F_{,t} + f\, X^3_{,t} = 0, \tag{23}$$

$$R_1 X^1_{,y} + R_2 \left(F_{,x} + X^3\right) = 0, \tag{24}$$

$$R_1 X^1_{,z} - xR_2 \left(F_{,x} + X^3\right) + f\, X^3_{,x} = 0, \tag{25}$$

where we have defined F as $F \equiv X^2 - xX^3$. Then we find that $det(R_{ab}) = R_0\, R_1\, R_2\, f$. Therefore, we will study the RCs according to whether $det(R_{ab}) = 0$ (degenerate case) or $det(R_{ab}) \neq 0$ (non-degenerate case).

If R_{ab} is non-degenerate, the standard results on KVs to deduce that the maximal dimension of the group of RCs in a pseudo-Riemannian manifold of dimension n is $n(n+1)/2$ are valid, and this occurs if and only if the Ricci tensor metric has constant curvature.

Thus, the maximal dimensions of the group of RCs for spacetimes are 10.

Therefore, the possible number of *proper* RCs for Bianchi type II spacetime are only $1, 2, 3, 4$ and 7.

Furthermore, in this study we will take the *proper* RCs to denote an RC which is not a KV, or HV, or SCKV.

For the degenerate R_{ab} of Bianchi II spacetime, we have the following possibilities:

(D0) all of the R_ℓ, $(\ell = 0, 1, 2))$ are zero;

(D1)-(D4) one of the R_ℓ and f are nonzero;

(D5)-(D10) two of the R_ℓ and f are nonzero;

(D11)-(D14) three of the R_ℓ and f are nonzero.

Case (D0) corresponds to the vacuum case in which every vector is a RC. Now, the solutions of the RC equations for Cases (D8), (D11) and (D14) are given the following.

The remaining degenerate cases are summarized in Table 1.

Case (D8). $R_i \neq 0,\ R_0 = 0 = f\ (i = 1, 2)$. In this case, we have

$$\mathbf{X} = -\frac{2R_1}{\alpha \dot{R}_1} F_{,y}\partial_t + (xF_{,y} + F_{,z})\partial_x + \left[F - xF_{,x} - \frac{x}{c_1} R_1^{1-\alpha}(xF_{,yy} + F_{,yz})\right]\partial_y - \left[F_{,x} + \frac{1}{c_1} R_1^{1-\alpha}(xF_{,yy} + F_{,yz})\right]\partial_z \qquad (26)$$

where $\dot{R}_1 \neq 0, R_2 = c_1 R_1^\alpha\ (\alpha \neq 0)$ and $F = F(x, y, z)$, and the following equations must be satisfied

$$x^2 F_{,yy} + F_{,zz} + 2xF_{,yz} = 0, \qquad (27)$$

$$F_{,xz} + xF_{,xy} + \frac{(\alpha - 1)}{\alpha} F_{,y} = 0. \qquad (28)$$

If R_1 and R_2 are arbitrary functions, then the RC vector is found as

$$\mathbf{X} = [A_1 + A_{1,x}]\,\xi_{(1)} - A_{1,x}\xi_{(2)} + a_0\xi_{(3)} \tag{29}$$

where a_0 is a constant, $A_1 = A_1(x)$, and $\xi_{(i)}$'s (i=1,2,3) are given in (6). When R_1 is arbitrary and $R_2 = c_2$ (constant), the following RC is obtained,

$$\mathbf{X} = -\frac{2R_1}{\dot{R}_1}A_{0,x}\partial_t + A_0\partial_x + [zA_0 + A_1 - x(zA_{0,x} + A_{1,x})]\,\partial_y - (zA_{0,x} + A_{1,x})\,\partial_z, \tag{30}$$

where $A_0 = A_0(x), A_1 = A_1(x)$, and $\dot{R}_1 \neq 0$.

Thus, we see that all subcases of this case give infinitely many RCs.

Table 1: The RCs of Bianchi II spacetimes in degenerate Ricci tensor cases. In this table c_1, c_2, c_3 are non-zero constants related with the components of Ricci tensor, a_0, a_1, a_2, a_3 are constants, and we have used the transformation $d\tau = \sqrt{R_0}\, dt$ in some cases.

Case	Constraint(s)	$\mathbf{X}$
D1	$R_0 \neq 0,\ R_i = 0 = f,$	$\frac{c}{\sqrt{R_0}}\partial_t + X^i(x^a)\partial_{x^i}, \quad (i = 1, 2)$
D2	$R_1 \neq 0,\ R_j = 0 = f,$ $\dot{R}_1 \neq 0,\ (j = 0, 2)$	$-\frac{2R_1}{\dot{R}_1} g(x)_{,x}\partial_t + g(x)\partial_x + X^\gamma(x^a)\partial_{x^\gamma},\ (\gamma = 2, 3)$
D3	$R_2 \neq 0,\ R_k = 0 = f,$ $\dot{R}_2 \neq 0,\ (k = 0, 1)$	$-\frac{2R_2}{\dot{R}_2} h_{,y}\partial_t + (xh_{,y} + h_{,z})\,\partial_x + (h - xh_{,x})\partial_y - h_{,x}\partial_z$ where $h = h(x, y, z)$.
D4	$f \neq 0,\ R_\ell = 0,$ $\dot{f} \neq 0,\ (\ell = 0, 1, 2)$	$-\frac{2f}{\dot{f}} g(z)_{,z}\partial_t + g(z)\partial_z + X^i(x^a)\partial_{x^i},\ (i = 1, 2)$
D5	$R_k \neq 0,\ R_2 = 0 = f$ $R_1 = c_1 e^{\beta\tau}$	$(a_0 x + a_1)\partial_\tau + \left[a_0\left(\frac{e^{-\beta\tau}}{\beta c_1} - \frac{\beta}{4}x^2\right) - \frac{\beta}{2}xa_1 + a_2\right]\partial_x + X^\gamma(x^a)\partial_{x^\gamma}$ where β is a constant.
D6	$R_j \neq 0,\ R_1 = 0 = f$ $R_2 = c_2 e^{\beta\tau}$ If R_0, R_2 are arbitrary,	$a_0\partial_\tau + \left(U_{0,z} - \frac{\beta}{2}a_0 x\right)\partial_x + \left(U_0 - xU_{0,x} - \frac{\beta}{2}a_0 y\right)\partial_y - U_{0,x}\partial_z$ where $U_0 = U_0(x, z)$. then $a_0 = 0$.

Case	Constraint(s)	$\mathbf{X}$
D7	$R_0 \neq 0 \neq f,\ R_i = 0$ $f = c_3 e^{\beta\tau}$	$(a_0 z + a_1)\partial_\tau + \left[a_0\left(\frac{e^{-\beta\tau}}{\beta c_3} - \frac{\beta}{4}z^2\right) - \frac{\beta}{2}a_1 z + a_2\right]\partial_z + X^i(x^a)\partial_{x^i}$
D9	$R_1 \neq 0 \neq f,\ R_j = 0$ $\dot{R}_1 \neq 0, f = \alpha_1 R_1$ $\alpha_1 h_{,xx} + h_{,zz} = 0$	$-\frac{2R_1}{\dot{R}_1}h_{,x}\partial_t + h\partial_x + X^2(x^a)\partial_y + \left(A_1 - \frac{1}{\alpha_1}\int h_{,z}dx\right)\partial_z$ where $A_1 = A_1(z)$ and $h = h(x,z)$.
	$R_1 = c_1, f = c_3$	$\left(a_1\frac{z}{c_1} + a_2\right)\partial_x + \left(-a_1\frac{x}{c_3} + a_3\right)\partial_z + X^j(x^a)\partial_{x^j}$
	$\dot{R}_1 \neq 0, f = \alpha_1 R_1^{\alpha_2}$	$-\frac{2a_1 R_1}{\alpha_2\dot{R}_1}\partial_t + \left(a_1\frac{x}{\alpha_2} + a_2\right)\partial_x + X^2(x^a)\partial_y + (a_1 z + a_3)\partial_z$
D10	$R_2 \neq 0 \neq f,\ R_k = 0$ $f = R_2^{1/\alpha},\ \dot{R}_2 \neq 0$	$-\frac{2\alpha R_2}{\dot{R}_2}g_{1,z}\partial_t + \left[(1-\alpha)xg_{1,z} - \alpha y g_{1,zz} + g_{2,z}\right]\partial_x$ $+\left(-\alpha y g_{1,z} + g_2\right)\partial_y - g_1\partial_z$ where $g_1 = g_1(z), g_2 = g_2(z)$ and $\alpha(\neq 0)$ is a constant.
	$R_2 = c_2,\ \dot{f} \neq 0$	$\frac{2f}{\dot{f}}g_{1,z}\partial_t + \left[xg_{1,z} + g_{2,z}\right]\partial_x + g_2\partial_y - g_1\partial_z$
D12	$R_j \neq 0 \neq f,\ R_1 = 0$ $\frac{R_{2,\tau}}{R_2} = 2\beta,\ \frac{f_{,\tau}}{f} = 2\alpha$	$a_0\partial_\tau + \left[a_0(\alpha-\beta)x + g(z)_{,z}\right]\partial_x$ $+\left[-a_0\beta y + g(z)\right]\partial_y + (-a_0\alpha z + a_1)\partial_z$

Case	Constraint(s)	$\mathbf{X}$
D13	$R_k \neq 0 \neq f,\ R_2 = 0$	
	$R_1 = c_1, f = c_2$	$(a_1 x + a_2 z + a_3)\partial_\tau + (-a_1 \frac{\tau}{c_1} + a_4)\partial_x + X^2(x^a)\partial_y + (-a_2 \frac{\tau}{c_2} + a_5)\partial_z$
		$(a_1 x + a_2)\partial_\tau + (-a_1 \frac{\tau}{c_1} + a_3 \frac{z}{c_1} + a_5)\partial_x + X^2(x^a)\partial_y + (-a_3 \frac{x}{c_2} + a_5)\partial_z$
		$(a_1 z + a_2)\partial_\tau + (a_3 \frac{z}{c_1} + a_4)\partial_x + X^2(x^a)\partial_y + (-a_1 \frac{\tau}{c_2} - a_3 \frac{x}{c_2} + a_4)\partial_z$
	$R_1 = c_1,\ f = c_2 e^{2\beta\tau}$	$(a_1 z + a_2)\partial_\tau + a_3\partial_x + X^2(x^a)\partial_y + \left[\frac{a_1}{2}\left(\frac{e^{-2\beta\tau}}{c_2\beta} - \beta z^2\right) - a_2\beta z + a_4\right]\partial_z$
	$R_1 = c_1 e^{2\alpha\tau},\ f = c_2$	$(a_1 x + a_2)\partial_\tau + \left[\frac{a_1}{2}\left(\frac{e^{-2\alpha\tau}}{c_1\alpha} - \alpha z^2\right) - a_2\alpha x + a_3\right]\partial_x + X^2(x^a)\partial_y + a_4\partial_z$
	$R_1 = c_1 e^{2\alpha\tau},\ f = c_2 e^{2\beta\tau}$	$a_1\partial_\tau + (-a_1\alpha x + a_2)\partial_x + X^2(x^a)\partial_y + (-a_1\beta z + a_3)\partial_z$
	$\left(\frac{R_{1,\tau}}{R_1}\right)_{,\tau} = \frac{2\alpha^2}{R_1}, \alpha^2 > 0$	$A_0\partial_\tau + \left[-\frac{R_{1,\tau}}{2\alpha^2 R_1}A_{0,x} - a_1 z \frac{f}{R_1} + a_2\right]\partial_x + X^2(x^a)\partial_y$
	$\left(\frac{f_{,\tau}}{f}\right)_{,\tau} = \frac{2\beta^2}{f},\ \beta^2 > 0$	$+\left[-\frac{f_{,\tau}}{2\beta^2 f}A_{0,z} + a_1 x + a_3\right]\partial_z$
	$f = -\frac{\beta^2}{\alpha^2}R_1 +$ const.	where $A_0 = \cosh(\alpha x)\left[a_4\cosh(\beta z) + a_5\sinh(\beta z)\right]$
		$+\sinh(\alpha x)\left[a_6\cosh(\beta z) + a_7\sinh(\beta z)\right]$

Case (D11). $R_i \neq 0 \neq f,\ R_0 = 0$. In this case, there exists an interesting situation where we have found the finite number of RCs in most of the subcases.

When $R_1 \neq c_1 R_2,\ f \neq c_3 R_1,\ \dot{R}_1 \neq 0$ *or* $R_1 = c_1 R_2, f \neq c_3 R_2$ *or* $R_1 = c_1, \dot{f} \neq 0$ *or* $R_1 = c_1,\ R_2 = c_2,\ \dot{f} \neq 0$, then the obtained RCs are *only KVs* given in (6).

In some subcases we have found the following proper RC in addition to the KVs given by (6)

$$\mathbf{X}_{(4)} = \epsilon_1\, z\partial_x + \frac{1}{2}\left(\epsilon_1\, z^2 - \epsilon_2 x^2\right)\partial_y - \epsilon_2 x\partial_z \tag{31}$$

where ϵ_1 and ϵ_2 are constants related with the appeared constraints.

If $R_2 = c_2 f = c_1 R_1, \dot{R}_1 \neq 0$ *or* $f = c_1 R_1$ and $R_2 = c_2$ *or* $R_1 = c_1$ and $f = c_2$ (these cases include *one proper RC* given the above), then the constants ϵ_1 and ϵ_2 take respectively the values $\epsilon_1 = c_1$ and $\epsilon_2 = c_2$ *or* $\epsilon_1 = c_1$ and $\epsilon_2 = 1$ *or* $\epsilon_1 = 1$ and $\epsilon_2 = c_1/c_2$, where c_1 and c_2 are nonzero constants.

When $R_1 = c_1,\ R_2 = c_2$ and $f = c_3$, we find *infinitely many RCs* as follows

$$\mathbf{X}_{(1)} = \xi_{(1)},\ \ \mathbf{X}_{(2)} = \xi_{(2)},\ \ \mathbf{X}_{(3)} = \xi_{(3)}$$

$$\mathbf{X}_{(4)} = X^0(x,y,z)\partial_t + \epsilon_1 z\partial_x + \frac{1}{2}\left(\epsilon_1 z^2 - \epsilon_2 x^2\right)\partial_y - \epsilon_2 x\partial_z, \tag{32}$$

where $\xi_{(1)}, \xi_{(2)}$ and $\xi_{(3)}$ are KVs given by (6); $\epsilon_1 = 1/c_1$ and $\epsilon_2 = 1/c_3$.

For the case $\dot{R}_1 \neq 0,\ R_1 = c_2 f^{1/(\beta-1)}$ and $R_2 = c_1 R_1^{\beta}$, it follows from the solution of RC equations (16)-(25) that the proper RC is

$$\mathbf{X}_{(4)} = -\frac{2R_1}{\dot{R}_1}\partial_t + x\partial_x + \beta y\partial_y + (\beta-1)z\partial_z \tag{33}$$

where β is a constant $(\neq 1, 2)$, and the Lie algebra is given by

$$\left[\mathbf{X}_{(1)}, \mathbf{X}_{(4)}\right] = \beta\mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] = \mathbf{X}_{(1)}, \tag{34}$$

$$\left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] = (\beta-1)\mathbf{X}_{(2)}, \quad \left[\mathbf{X}_{(3)}, \mathbf{X}_{(4)}\right] = \mathbf{X}_{(3)}. \tag{35}$$

When $\beta = 1$, i.e., $R_2 = c_1 R_1,\ f = c_3 R_1$ and $\dot{R}_1 \neq 0$, the number of RCs becomes infinite which are the following ones

$$X^0 = -\frac{2R_1}{\dot{R}_1}F_{,y}, \quad X^1 = G, \tag{36}$$

$$X^2 = F - xF_{,x} - \frac{x}{c_1}G_{,y},\ X^3 = -F_{,x} - \frac{1}{c_1}G_{,y} \tag{37}$$

where $F = F(x,y,z)$, $G = G(x,y,z)$, and the following constraint equations have to be satisfied

$$G_{,z} + xG_{,y} - \frac{k}{c_1}\left(F_{,yy} + c_1 F_{,xx}\right) = 0, \tag{38}$$

$$G - xF_{,y} - F_{,z} + \frac{k}{c_1^2}\left(G_{,yy} + c_1 F_{,xy}\right) = 0, \tag{39}$$

$$xG_{,yy} + G_{,yz} + c_1\left(F_{,xz} + xF_{,xy} + F_{,y}\right) = 0. \tag{40}$$

For $R_1 = c_2 f$ and $R_2 = c_1 R_1^2$, i.e. $\beta = 2$, the obtained proper RCs are given by

$$\mathbf{X}_{(4)} = -\frac{2R_1}{\dot{R}_1}\partial_t + x\partial_x + 2y\partial_y + z\partial_z,$$

$$\mathbf{X}_{(5)} = \epsilon_1 z\partial_x + \frac{1}{2}\left(\epsilon_1 z^2 - \epsilon_2 x^2\right)\partial_y - \epsilon_2 x\partial_z, \tag{41}$$

where $\epsilon_1 = 1$ and $\epsilon_2 = c_2$. The corresponding Lie algebra has the following non-vanishing commutators:

$$\begin{aligned}
&\left[\mathbf{X}_{(1)}, \mathbf{X}_{(4)}\right] = 2\mathbf{X}_{(1)}, \\
&\left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] = \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] = \mathbf{X}_{(2)}, \\
&\left[\mathbf{X}_{(3)}, \mathbf{X}_{(4)}\right] = \mathbf{X}_{(3)}, \quad \left[\mathbf{X}_{(3)}, \mathbf{X}_{(5)}\right] = -c_2\mathbf{X}_{(2)}.
\end{aligned} \tag{42}$$

If $R_1 = c_1$ and $R_2 = af$, where a is a constant, then we get

$$\mathbf{X}_{(4)} = -\frac{2R_2}{\dot{R}_2}\partial_t + y\partial_y + z\partial_z, \tag{43}$$

where $\dot{R}_2 \neq 0$, and the Lie algebra is given by

$$\left[\mathbf{X}_{(1)}, \mathbf{X}_{(4)}\right] = \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] = \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] = \mathbf{X}_{(2)}. \tag{44}$$

If $R_1 = b/f$ and $R_2 = c_2$, where b is a constant, then we have

$$\mathbf{X}_{(4)} = -\frac{2R_1}{\dot{R}_1}\partial_t + x\partial_x - z\partial_z, \tag{45}$$

where $\dot{R}_1 \neq 0$, and the Lie algebra has the following non-vanishing commutators

$$\left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] = \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] = \mathbf{X}_{(2)}. \tag{46}$$

Thus, we have *finite* number of RCs in most of the subcases of this case even if the R_{ab} is degenerate.

Case (D14). $R_\ell \neq 0,\ f = 0$ $(\ell = 0, 1, 2)$. In this case, we have the following RCs

$$\mathbf{X} = U_0 \partial_x + \left[z U_0 + U_1 - x \left(z U_{0,x} + U_{1,x} \right) \right] \partial_y - \left(z U_{0,x} + U_{1,x} \right) \partial_z$$

where $U_0 = U_0(x), U_1 = U_1(x)$.

If $R_1 = \frac{-1}{2\alpha\tau}$ and $R_{2,\tau} \neq \frac{-\beta}{\alpha\tau} R_2$, then we get

$$\mathbf{X} = a_1 \partial_x + \left[a_1 z + U_0 - x U_{0,x} \right] \partial_y - U_{0,x} \partial_z .$$

When $R_1 = \frac{-1}{2\alpha\tau}$ and $R_2 = c_2 e^{-\beta/\alpha}$, the RC vector field becomes

$$\mathbf{X} = a_1 \partial_\tau + (-a_1 \alpha x + a_2) \partial_x - \left[a_1 (\alpha - \beta) z - U_{0,x} \right] \partial_z + \left(-a_1 \beta y + a_2 z + U_0 - x U_{0,x} \right) \partial_y .$$

Thus, the number of RCs is infinite in this case.

4. Ricci Collineations for Non-degenerate Ricci Tensor Cases

Now, we consider the RCs in non-degenerate cases, i.e. $det(R_{ab}) \neq 0$, admitted by Bianchi II spacetime. We define a set $\{\dot{R}_\ell(t), \dot{f}(t)\}$ for functions R_ℓ and f, where $\ell = 0, 1, 2$. Thus, we have the following possibilities:

(ND1)-(ND4) three elements of the set are zero;

(ND5)-(ND10) two elements of the set are zero;

(ND11)-(ND14) one element of the set is zero;

(ND15) all elements of the set are zero.

Before giving the solutions of these cases, we write the constraint equations appearing in the classification as

$$R_1 \left(\frac{R_{1,\tau}}{2R_1} \right)_{,\tau} = \epsilon\, \alpha^2, \quad R_2 \left(\frac{R_{2,\tau}}{2R_2} \right)_{,\tau} = \epsilon\, \beta^2, \tag{47}$$

where ϵ takes values 1 or c_0 (a constant related with R_0), and α, β are separation constants. If $\epsilon = 1$, then we use the transformation $d\tau = \sqrt{R_0}\, dt$. Otherwise, i.e. when $\epsilon = c_0$, then R_0 becomes a constant ($= c_0$), and the conformal time τ is equivalent to the physical time t. If α^2 and β^2 are zero, then it follows from the constraint equations (47) that $R_1 = c_1 e^{2\eta\,\tau}$ and $R_2 = c_2 e^{2\mu\,\tau}$, where η and μ are integration constants.

The obtained solutions of these possible cases are given Table 2.

Table 2: The RCs of Bianchi II spacetimes in non-degenerate Ricci tensor cases. Here, we have used c_0, c_1, c_2, c_3 as the non-zero constants related with the components of Ricci tensor.

Case	Constraints	$\sharp$ of RCs	proper RCs
ND1	$\dot{R}_0 \neq 0,\ R_1 = c_1,\ R_2 = c_2,\ f = c_3,$ $C^2 \neq \frac{c_3}{c_1} A^2$	5	$\mathbf{X}_{(4)} = \frac{z}{c_1}\partial_x + \frac{1}{2}\left(\frac{z^2}{c_1} - \frac{x^2}{c_3}\right)\partial_y - \frac{x}{c_3}\partial_z,$ $\mathbf{X}_{(5)} = \partial_\tau$
ND2	$\dot{R}_1 \neq 0,\ R_0 = c_0,\ R_2 = c_2,\ f = c_3$	3	—
ND3	$\dot{R}_2 \neq 0\ R_0 = c_0,\ R_1 = c_1,\ f = c_3$	4	$\mathbf{X}_{(4)}$ is same as ND1
ND4	$\dot{f} \neq 0,\ R_0 = c_0,\ R_1 = c_1,\ R_2 = c_2$	3	—
ND5	$\dot{R}_k \neq 0,\ R_2 = c_2,\ f = c_3\ (k = 0, 1)$	3	—
ND6	$\dot{R}_j \neq 0,\ R_1 = c_1\ f = c_3\ (j = 0, 2)$	4	$\mathbf{X}_{(4)}$ is same as ND1
ND7	$\dot{R}_0 \neq 0 \neq \dot{f},\ R_1 = c_1,\ R_2 = c_2$	3	—
ND8	$\dot{R}_i \neq 0,\ R_0 = c_0,\ f = c_3\ (i = 1, 2)$	3	—
ND9	$\dot{R}_1 \neq 0 \neq \dot{f},\ R_0 = c_0,\ R_2 = c_2$	4	$\mathbf{X}_{(4)}$ is same as ND1
ND10	$\dot{R}_2 \neq 0 \neq \dot{f},\ R_0 = c_0,\ R_1 = c_1$	3	—

Case	Constraints	♯ of RCs	proper RCs
ND11	$\dot{R}_j \neq 0 \neq \dot{f},\ R_0 = c_0 = \epsilon,\ \alpha^2 = 0 = \beta^2$		$\mathbf{X}_{(4)} = \frac{z}{c_1}\partial_x + \frac{1}{2}\left(\frac{z^2}{c_1} - \frac{x^2}{c_3}\right)\partial_y - \frac{x}{c_3}\partial_z,$
	$R_1 = c_1 e^{2\eta t},\ R_2 = c_2 e^{2\eta t},\ f = c_3 e^{2\eta t}, \mu = 2\eta$	5	$\mathbf{X}_{(5)} = \partial_t - \eta\, x\, \partial_x - 2\eta\, y\, \partial_y - \eta\, z \partial_z$
	$R_1 = c_1 e^{2\eta t},\ R_2 = c_2 e^{2\mu t},\ f = c_3 e^{2\eta t},\ \mu \neq 2\eta$	4	$\mathbf{X}_{(4)}$ is same as ND1
ND12	$\dot{R}_0 \neq 0,\ R_1 = c_1,\ R_2 = c_2 e^{2\mu\tau},\ f = c_3 e^{2\mu\tau}$ $\epsilon = 1,\ \beta^2 = 0$	4	$\mathbf{X}_{(4)} = \partial_\tau - \mu\, y\, \partial_y - \mu\, z \partial_z$
ND13	$\dot{R}_0 \neq 0,\ R_1 = c_1 e^{2\eta\tau},\ R_2 = c_2,\ f = c_3 e^{2\eta\tau}$ $\epsilon = 1,\ \alpha^2 = 0$	4	$\mathbf{X}_{(4)}$ is same as ND1
ND14	$\dot{R}_\ell \neq 0,\ f = c_3\ (\ell = 0, 1, 2))$	3	—
ND15	$R_0 = c_0,\ R_1 = c_1,\ R_2 = c_2,\ f = c_3$	5	$\mathbf{X}_{(4)}$ is same as ND1 $\mathbf{X}_{(5)} = \partial_t$

For the cases (ND2), (ND4), (ND5), (ND7), (ND8), (ND10) and (ND14), we have only the KVs.

In cases (ND3), (ND6) and (ND9), and some subcases of (ND11; where $R_0 = c_0 = \epsilon,\ \alpha^2 = 0 = \beta^2,\ f = c_3 e^{2\eta\, t}, \mu \neq 2\eta$) and (ND13; where $\epsilon = 1,\ \alpha^2 = 0, R_2 = c_2,\ f = c_3 e^{2\eta\, \tau}$), we have only *one* proper RC

$$\mathbf{X}_{(4)} = \frac{z}{c_1}\partial_x + \frac{1}{2}\left(\frac{z^2}{c_1} - \frac{x^2}{c_3}\right)\partial_y - \frac{x}{c_3}\partial_z \tag{48}$$

where c_1 and c_3 are non-zero constants, and the Lie algebra has the following non-vanishing commutators

$$\left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] = \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] = \frac{1}{c_1}\mathbf{X}_{(3)}, \quad \left[\mathbf{X}_{(3)}, \mathbf{X}_{(4)}\right] = -\frac{1}{c_3}\mathbf{X}_{(2)}. \tag{49}$$

For the case (ND12; where $\epsilon = 1,\ \beta^2 = 0,\ R_1 = c_1,\ f = c_3 e^{2\mu\, \tau}$), the obtained proper RC is given as

$$\mathbf{X}_{(4)} = \partial_\tau - \mu\, y\, \partial_y - \mu\, z \partial_z \tag{50}$$

with the non-vanishing commutators

$$\left[\mathbf{X}_{(1)}, \mathbf{X}_{(4)}\right] = -\mu \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] = \mathbf{X}_{(1)}, \quad \left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] = -\mu \mathbf{X}_{(2)}. \tag{51}$$

In the cases (ND1), (ND11; where $R_0 = \epsilon = c_0,\ \alpha^2 = 0 = \beta^2,\ f = c_3 e^{2\eta\, t},\ \mu = 2\eta$), and (ND15), we have found *two* proper RCs. For the cases (ND1) and (ND15), one of these proper RCs is given by (48) and the other one is $\mathbf{X}_{(5)} = \partial_\tau$ or $\mathbf{X}_{(5)} = \partial_t$, respectively. For the case (ND11), in addition to the fourth RC given by (48), the fifth proper RC is obtained as

$$\mathbf{X}_{(5)} = \partial_t - \eta\, x\, \partial_x - 2\eta\, y\, \partial_y - \eta\, z \partial_z \qquad (52)$$

where $\eta = \mu/2$. For the last case, the non-vanishing commutators of the Lie algebra are given by

$$\begin{aligned} \left[\mathbf{X}_{(1)}, \mathbf{X}_{(5)}\right] &= -2\eta \mathbf{X}_{(1)}, \quad & \left[\mathbf{X}_{(2)}, \mathbf{X}_{(3)}\right] &= \mathbf{X}_{(1)}, \\ \left[\mathbf{X}_{(2)}, \mathbf{X}_{(4)}\right] &= \frac{1}{c_1} \mathbf{X}_{(3)}, \quad & \left[\mathbf{X}_{(2)}, \mathbf{X}_{(5)}\right] &= -2\eta \mathbf{X}_{(2)}, \\ \left[\mathbf{X}_{(3)}, \mathbf{X}_{(4)}\right] &= -\frac{1}{c_3} \mathbf{X}_{(2)}, \quad & \left[\mathbf{X}_{(3)}, \mathbf{X}_{(5)}\right] &= -\eta \mathbf{X}_{(3)}. \end{aligned} \qquad (53)$$

a. In this study, we have solved the RC Eqs.(16)-(25) for Bianchi II spacetime (5), and obtained all possible RCs according to the degenerate or non-degenerate Ricci tensor. We have found that if the Ricci tensor is degenerate, section 3, then there are many cases of RCs for the Bianchi II spacetime with infinite degrees of freedom except for most of the subcases of case (D11), the groups of RCs are finite dimensional, in which there are *one* or *two* proper RCs. When the Ricci tensor is non-degenerate, section 4, we have obtained finite number of RCs which are *three, four* and *five*. Therefore, the number of *proper* RCs in non-degenerate Ricci tensor cases are *one* or *two*. In some cases of sections 3 and 4, the results are given in terms of R_0 and some integration constants together with differential constraints related to the components $R_1, R_2,$ and f which must be satisfied.

Also, in any case of degenerate or non-degenerate cases of the Ricci tensor, we have also obtained different constraint equations. When we could solve these constraint equations, it could be able to find new exact solutions of EFEs.

b. Before this section, we have not used any form of the energy-momentum tensor T_{ab}. As an application, for the Bianchi II spacetime, we consider the perfect fluid, which is given by $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where u_a is the four velocity of the normalized fluid, ρ and p are the energy density and the pressure, respectively. Therefore, if the universe is filled with a perfect fluid, then using the EFEs one can obtain that

$$R_0 = \frac{1}{2}(\rho + 3p), \tag{54}$$

$$R_1 = \frac{A^2}{2}(\rho - p), \quad R_2 = \frac{B^2}{2}(\rho - p), \quad f = \frac{C^2}{2}(\rho - p). \tag{55}$$

The linear form of a barotropic equation of state $p = p(\rho)$ is given by

$$p = w\rho \equiv (\gamma - 1)\rho, \tag{56}$$

where ρ is the energy density, p is the pressure, and w (and γ) is a constant. Causality then requires w to be in the interval $-1 \leq w \leq 1$. Hence the parameters $w = -1, 0, 1/3$, and 1 correspond to *vacuum fluid, dust filled universe, radiation* and *stiff matter*, respectively.

The mathematical instability of the parameter w might lead to some interesting physics. The matter with the property $\rho > 0$ but $p < -\rho < 0$ (i.e., $w < -1$) is dubbed the phantom energy. For the behavior of the matter in the quintessence regime, the interval of state parameter w is $-1 < w < -1/3$. It ought to be noted that both quintessence and phantom fluids lead to the inequality $\rho + 3p \leq 0$, thus breaking the strong energy condition.

Now, using (54) and (55) in the metric (15), the Ricci tensor metric of perfect fluid Bianchi II spacetime becomes

$$2ds^2_{Ric} = (\rho + 3p)\, dt^2 + (\rho - p) \left[dx^2 + (dy - xdz)^2 + dz^2\right], \qquad (57)$$

which has same signature with the metric (5) if $w = -1$, that is, $p = -\rho$. For the latter case, there is a relation between the generic Bianchi II metric (5) and the Ricci tensor metric (57) such as $ds^2_{Ric} = \rho\, ds^2$, that is, the spacetimes (5) and (57) are conformally related with the conformal factor ρ. Beside the phantom barrier $w = -1$, both the phantom region $w < -1$ and quintessence region, $-1 < w < -1/3$, give rise to the Lorentzian signature metrics. But in the causal region $-1 \leq w \leq 1$, it is interesting to note that we have the Euclidean signature metric for the interval $-1/3 < w < 1$.

The matter tensor metric of Bianchi II spacetime has the form

$$ds^2_{Matter} \equiv T_{ab} dx^a dx^b = T_0 dt^2 + T_1 dx^2 + T_2 (dy - x dz)^2 + f_M dz^2, \tag{58}$$

where T_{ab} is given by

$$T_{00} \equiv T_0 = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{B^2}{4A^2C^2}, \tag{59}$$

$$T_{11} \equiv T_1 = -A^2 \left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{B^2}{4A^2C^2} \right), \tag{60}$$

$$T_{22} \equiv T_2 = -B^2 \left(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{3B^2}{4A^2C^2} \right), \tag{61}$$

$$T_{23} = -x T_2, \quad T_{33} \equiv T_3 = f(t)_M + x^2 T_2, \tag{62}$$

where $f(t)_M$ is defined as

$$f(t)_M = -C^2 \left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{B^2}{4A^2C^2} \right). \tag{63}$$

For the perfect fluid Bianchi II spacetime, we have $T_0 = \rho, T_1 = pA^2, T_2 = pB^2$ and $f_M = pC^2$, which yields the following metric

$$ds^2_{Perfect} = \rho\, dt^2 + p \left[A^2 dx^2 + B^2 (dy - x dz)^2 + C^2 dz^2 \right]. \tag{64}$$

Therefore, using (59)-(63), the energy density and the pressure become

$$\rho = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} - \frac{B^2}{4A^2C^2}, \quad (65)$$

$$p = -\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{B^2}{4A^2C^2}\right), \quad (66)$$

$$= -\left(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{3B^2}{4A^2C^2}\right), \quad (67)$$

$$= -\left(\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{B^2}{4A^2C^2}\right). \quad (68)$$

Hence the curvature scalar given in (14) is obtained as $R = \rho - 3p$ and $\rho + 3p = 2R_0$.

The matter tensor metric is positive-definite when $\rho > 0$ and $p > 0$. For the perfect fluid the energy conditions are given as [13]

$$p = p(\rho), \quad \rho > 0, \quad 0 \leq p \leq \rho. \quad (69)$$

Under the assumption of perfect fluid in non-degenerate cases of the energy-momentum tensor for Bianchi II spacetime, we have obtained from the constraints related with each of the non-degenerate cases that $R = (1 - 3w)\rho$. Thus, if $w = 1/3$ (radiation filled universe), then the curvature scalar R of perfect fluid Bianchi II spacetime vanishes.

c. In degenerate cases of T_{ab} for perfect fluid, i.e. when $det(T_{ab}) = 0$, the energy conditions for the case ($T_0 \neq 0,\ T_i = 0 = f_M,\ i = 1, 2$) are satisfied but for remaining ones are not. In the latter case of degenerate T_{ab}, we have *dust* ($p = 0$) and the curvature scalar has the form $R = \rho$, i.e. $R > 0$.

In same case of the degenerate R_{ab}, (D1;$R_0 \neq 0,\ R_i = 0 = f$) filled with perfect fluid, we found that $p = \rho = \frac{1}{2}R_0$ (*stiff fluid*) and $R = -2\rho$, that is, $R < 0$. This is an example of difference of the results obtained from RCs and MCs in degenerate case.

Mathematical similarities between the R_{ab} and T_{ab} mean many techniques for their study should show some similarities. When the T_{ab} (equivalently G_{ab}) is non-degenerate, the determination of MCs for the Bianchi II spacetime follows immediately from Table 2 without any further calculations.

Then we will give an example for differences of the obtained results from RCs and MCs in non-degenerate case. In case (ND15;$\dot{R}_\ell = 0 = f$, $\ell = 0, 1, 2$) of non-degenerate Ricci tensor, assuming the perfect fluid for Bianchi II spacetime, we found that $\rho + 3p = const$. But the corresponding case ($\dot{T}_\ell = 0 = f_M$) of MCs in perfect fluid Bianchi II metric yields that $\rho + 3p \neq$ const.

References

[1] G. H. Katzin, J. Levine, and W. R. Davis, *J. Math. Phys.* **10** 617 (1969).

[2] D. Kramer, H. Stephani, M. A. H. MacCallum, and E. Herlt, *Exact Solutions of Einstein Field Equations*, (Cambridge Univ. Press, Cambridge, 1980).

[3] U. Camci and A. Barnes, *Class. Quant. Grav.***19** 393 (2002).

[4] M. Tsamparlis, and P. S. Apostolopoulos, *J. Math. Phys.***41** 7543 (2000).

[5] U. Camci, İ. Yavuz, H. Baysal, İ. Tarhan, and İ. Yılmaz, *Int. J. Mod. Phys. D***10** 751 (2001).

[6] U. Camcı and İ. Yavuz, *Int. J. Mod. Phys. D***12** 89 (2003).

[7] İ. Yavuz, and U. Camci, *Gen. Rel. Grav.* **28** 691 (1996).

[8] M. Tsamparlis and P. S. Apostolopoulos, *Gen. Rel. Grav.* **36** 47 (2004).

[9] U. Camci and İ. Turkyilmaz, *Gen. Rel. Grav.***36** 2005 (2004).

[10] J. Carot, J. da Costa, and E. G. L. R. Vaz, *J. Math. Phys.* **35** 4832 (1994).

[11] G. S. Hall, I. Roy, and E. G. L. R. Vaz, *Gen. Rel. Grav.* **28** 299 (1996).

[12] S. R. Roy and S. K. Banerjee, *Class.Quantum Grav.* **14** 2845 (1997).

[13] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structures of Space-time*, (Cambridge Univ. Press, Cambridge, 1973).